Meijer's G – Function – Application to Stereographic Semicircular Half Logistic Distribution

Y. Sreekanth, S.V.S. Girija

[1]Lecturer in Mathematics,[2]Professor of Mathematics
[12]Department of Mathematics,
[12]Hindu College, Guntur, India

Abstract - The object of this paper is to figure out the application of Meijer's G- function in Circular Statistics, in particular, the derivation and evaluation of trigonometric moments. Here, the characteristic function and trigonometric moments of Stereographic Semicircular Half Logistic distribution are derived using Meijer's G – function. Population characteristics of Stereographic Semicircular Half Logistic distribution are also evaluated applying trigonometric moments.

Index Terms - Meijer's G – function, semicircular distributions, characteristic function, trigonometric moments, population characteristics

I. Introduction

The Meijer G - function is a very general function which reduces to simpler special functions in many common cases. The probability density functions of products of independent beta, gamma and central gaussian random variables are shown to be ***Meijer G-functions***.

Stereographic Semicircular Half Logistic distribution (**SSCHLD**) over a semicircular segment in natural phenomenon is developed in [9]. The density, distribution and certain population characteristics are derived for the proposed semicircular model. The characteristic function of Stereographic Semicircular Half Logistic distribution is discussed and the first two trigonometric moments are figured out for proposed model. The concept of Meijer's *G*-function is adopted to derive first two trigonometric moments of Stereographic Semicircular Half Logistic distribution.
Program listing developed in MATLAB to draw graphs and compute population characteristics are included.

II. Literature Survey

It is evident that Meijer's G – function to distribution problems in Statistics is applied in [2]. Pishkoo and Darus in [7] established Meijer G-Functions as Solutions of Differential Equations in Physical Models. The generalized hypergeometric function as the Meijer *G*-function is presented in [3]. In [8] Qureshi contributed Generalizations and applications of Srinivasa Ramanujan's integral associated with infinite Fourier sine transforms in terms of Meijer's *G*-function.

III. Stereographic Semicircular Half Logistic distribution

A random variable *X* on the real line is said to have a Half Logistic distribution with location parameter α and scale parameter $\beta > 0,$ if the probability density function and probability distribution function of X are given respectively by

$$f(x) = \frac{2}{\beta}\left[1+\exp\left(\frac{-(x-\alpha)}{\beta}\right)\right]^{-2}\exp\left(\frac{-(x-\alpha)}{\beta}\right), \qquad 0 < x < \infty,\ \beta > 0. \qquad (3.1)$$

$$F(x) = \frac{1-\exp\left(\frac{-(x-\alpha)}{\beta}\right)}{1+\exp\left(\frac{-(x-\alpha)}{\beta}\right)},\ 0 < x < \infty \qquad (3.2)$$

Then by applying modified inverse stereographic projection well-defined by a one-to-one mapping $x = v\tan\left(\frac{\theta}{2}\right)$, $v \in R^{+}$, this leads to a semicircular model on unit semicircle. Stereographic Semicircular Half Logistic distribution is defined in [9] by applying inverse stereographic projection on respective linear model on the lines of [6] as follows

Definition 3.1:

A random variable X_{SC} on the semicircle is said to have the Stereographic Semicircular Half Logistic distribution with location parameter μ scale parameter $\sigma > 0$ denoted by S**SCHLD** (σ, μ) , if the probability density and the cumulative distribution functions are respectively given by

$$g(\theta)=\frac{1}{\sigma}\sec^2\left(\frac{\theta}{2}\right)\left[1+\exp\left(\frac{-\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right]^{-2}\exp\left(-\left(\frac{\tan\left(\frac{\theta}{2}\right)-\mu}{\sigma}\right)\right), \quad (3.3)$$

$0<\theta<\pi,\ \sigma=\frac{\beta}{\nu}>0,\ \mu=\frac{\alpha}{\nu}$

$$G(\theta)=\left[1-\exp\left(\frac{-\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right]\left[1+\exp\left(\frac{-\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right]^{-1},\ 0<\theta<\pi,\ \sigma=\frac{\beta}{\nu}>0,\ \mu=\frac{\alpha}{\nu} \quad (3.4)$$

Hence the proposed new model S**SCHLD** (σ,μ) is a semicircular model.

IV CHARACTERISTIC FUNCTION OF STEREOGRAPHIC SEMICIRCULAR HALF LOGISTIC DISTRIBUTION

The characteristic function of a Semicircular model with probability density function $g(\theta)$ is defined as $\varphi_p(\theta)=\int_0^{\pi} e^{ip\theta} g(\theta)d\theta\ , p\in Z$. The characteristic function of a Stereographic circular model can be derived in terms of respective linear model. In [4] Lukacs demonstrated the hypothesis connected with the characteristic function of linear model which is applied here on account of Stereographic semicircular models. On these lines the characteristic function of a Stereographic Semicircular model is proposed.

Theorem 4.1: If $G(\theta)$ and $g(\theta)$ are the cdf and the pdf of the Stereographic Semicircular model and $F(x)$ and $f(x)$ are the cdf and the pdf of the respective linear model, then the characteristic function of Stereographic Semicircular model is $\varphi_{X_{SC}}(p)=\varphi_{2\tan^{-1}\left(\frac{x}{\nu}\right)}(p), p\in Z$

Analytical derivation of this integral is made possible by using the concept of Meijer's G – function and first two trigonometric moments which play vital role in evaluation of population characteristics are obtained in terms of Meijer's G – function and here MATLAB tools are employed to evaluate the values of the characteristic function. Making use of computed values, the graphs for real and imaginary parts of the characteristic function are plotted here.

Meijer's G-function:

$$\int_0^{\infty} x^{2\nu-1}\left(u^2+x^2\right)^{Q-1} e^{-\mu x}dx=\frac{u^{2\nu+2Q-2}}{2\sqrt{\pi}\Gamma(1-Q)}G_{13}^{31}\left(\frac{\mu^2u^2}{4}\middle|\begin{matrix}1-\nu\\ 1-Q-\nu,0,\frac{1}{2}\end{matrix}\right) \quad (4.1)$$

for $\left|\arg u\pi\right|<\frac{\pi}{2}, \operatorname{Re}\mu>0$ and $\operatorname{Re}\nu>0$ and $G_{13}^{31}\left(\frac{\mu^2u^2}{4}\middle|\begin{matrix}1-\nu\\ 1-Q-\nu,0,\frac{1}{2}\end{matrix}\right)$ is called as Meijer's **G**-function [1], formula no. 3.389.2).

The characteristic function of Stereographic Semicircular Half Logistic Model is

$$\varphi_{X_{SC}}(p)=\int_0^{\pi} e^{ip\theta}g(\theta)d\theta=\int_0^{\pi} e^{ip\theta}\frac{1}{\sigma}\sec^2\left(\frac{\theta}{2}\right)\left[1+\exp\left(\frac{-\left(\tan\left(\frac{\theta}{2}\right)-\mu\right)}{\sigma}\right)\right]^{-2}\exp\left(-\left(\frac{\tan\left(\frac{\theta}{2}\right)-\mu}{\sigma}\right)\right)d\theta$$

$0<\theta<\pi,\ \sigma>0,\ \mu<\tan\left(\frac{\theta}{2}\right)$ (4.2)

The graphs of the characteristic function for various values of parameter are plotted here

Fig 1 Graph of characteristic function of Stereographic Semicircular Half Logistic Distribution for $\sigma = 0.5$

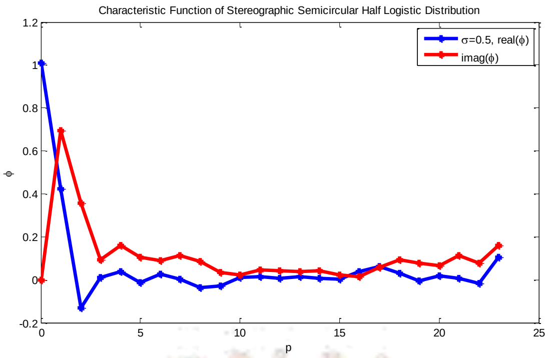
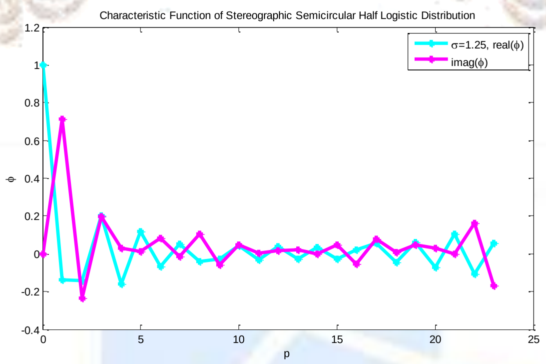


Fig 2 Graph of characteristic function of Stereographic Semicircular Half Logistic Distribution for $\sigma = 1.25$



V. TRIGONOMETRIC MOMENTS OF THE STEREOGRAPHIC SEMICIRCULAR HALF LOGISTIC DISTRIBUTION

The trigonometric moments of the distribution are specified by $\{\varphi_p : p = \pm1, \pm2, \pm3, ...\}$, where $\varphi_p = \alpha_p + i\beta_p$, with $\alpha_p = E(\cos p\theta)$ and $\beta_p = E(\sin p\theta)$ being the pth order cosine and sine moments of the random angle θ respectively.

Theorem 5.1 With the probability density function of Stereographic Semicircular Half Logistic distribution with μ=0, the first two trigonometric moments are

$$\alpha_1 = 1 - \frac{2}{\sigma\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n (n+1) G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix} \right),$$

$$\beta_1 = \frac{2}{\sigma\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n (n+1) G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix} \right),$$

$$\alpha_2 = 1 + \frac{2}{\sigma\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n (n+1) \left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -\frac{3}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix} \right) - G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix} \right) \right],$$

$$\beta_2 = \frac{4}{\sigma\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n (n+1) \left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix} \right) - 4G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -1 \\ 0, 0, \frac{1}{2} \end{matrix} \right) \right].$$

The first and second trigonometric moments are enough for calculating population characteristics.

Proof:

$$\varphi_p = \int_0^\pi e^{ip\theta} g(\theta)d\theta = \int_0^\pi \cos(p\theta)g(\theta)d\theta + i\int_0^\pi \sin(p\theta)g(\theta)d\theta$$

To derive the first cosine moment

$$\alpha_1 = \frac{1}{\sigma}\int_0^\pi \cos\theta \sec^2\left(\frac{\theta}{2}\right)\left[1+e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)}\right]^{-2} e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)} d\theta,$$ we use the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\cos(\theta) = 1 - \frac{2x^2}{1+x^2}$

and the above integral formula

$$\alpha_1 = \frac{2}{\sigma}\int_0^\infty \left[1-\frac{2x^2}{1+x^2}\right]\left[1+e^{-\frac{x}{\sigma}}\right]^{-2} e^{-\frac{x}{\sigma}} dx$$

$$= 1 - \frac{4}{\sigma}\sum_{n=0}^\infty (-1)^n (n+1)\int_0^\infty x^{2(\frac{3}{2})-1}(1+x^2)^{0-1} e^{-\left(\frac{n+1}{\sigma}\right)x} dx$$

(by substituting $u = 1, \nu = \frac{3}{2}, Q = 0, \mu = \frac{n+1}{\sigma}$ in (3.1)

$$\alpha_1 = 1 - \frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^\infty (-1)^n (n+1) G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix}\right)$$

To derive the first sine moment $\beta_1 = \frac{1}{\sigma}\int_0^\pi \sin\theta \sec^2\left(\frac{\theta}{2}\right)\left[1+e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)}\right]^{-2} e^{-\frac{1}{\sigma}\left(\tan\left(\frac{\theta}{2}\right)\right)} d\theta$, we use the transformation $x = \tan\left(\frac{\theta}{2}\right)$, $\sin(\theta) = \frac{2x}{1+x^2}$ and result follows by the same integral formula of α_1.

$$\beta_1 = \frac{4}{\sigma}\int_0^\infty \left[\frac{x}{1+x^2}\right]\left[1+e^{-\frac{x}{\sigma}}\right]^{-2} e^{-\frac{x}{\sigma}} dx$$

$$= \frac{4}{\sigma}\sum_{n=0}^\infty (-1)^n (n+1)\int_0^\infty x^{2(1)-1}(1+x^2)^{0-1} e^{-\left(\frac{n+1}{\sigma}\right)x} dx$$

$$\beta_1 = \frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^\infty (-1)^n (n+1) G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix}\right)$$

To obtain second cosine and sine moments α_2 and β_2, we use the transformations $x = \tan\left(\frac{\theta}{2}\right)$, $\cos 2\theta = 1 + \frac{8x^4}{(1+x^2)^2} - \frac{8x^2}{(1+x^2)}$ and $\sin 2\theta = \frac{4x}{(1+x^2)} - \frac{8x^3}{(1+x^2)^2}$, the results of α_2 and β_2 follows by the same integral formula of α_1.

$$\alpha_2 = 1 + \frac{2}{\sigma\sqrt{\pi}}\sum_{n=0}^\infty (-1)^n (n+1)\left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -\frac{3}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix}\right) - G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -\frac{1}{2} \\ -\frac{1}{2}, 0, \frac{1}{2} \end{matrix}\right)\right],$$

$$\beta_2 = \frac{4}{\sigma\sqrt{\pi}}\sum_{n=0}^\infty (-1)^n (n+1)\left[G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} 0 \\ 0, 0, \frac{1}{2} \end{matrix}\right) - 4G_{13}^{31}\left(\left(\frac{n+1}{2\sigma}\right)^2 \middle| \begin{matrix} -1 \\ 0, 0, \frac{1}{2} \end{matrix}\right)\right].$$

Higher-order moments can be obtained similarly.

The formulae for population characteristics for circular distributions are available in [5]. These characteristics for the Stereographic Semicircular Half Logistic distribution are also figured out using their respective trigonometric moments and can be stated in terms of trigonometric moments α_p and β_p which are presented here.

Table 1 Population Characteristics of SSCHLD

	$\sigma = 0.25$	$\sigma = 0.5$	$\sigma = 0.75$	$\sigma = 1.25$
Trigonometric Moments				
α_1	0.7962	0.2571	0.1003	-0.1801
α_2	0.4310	-0.4301	-0.3403	-0.1897
β_1	0.3774	0.8297	0.7931	0.7390
β_2	0.4235	0.3395	0.0737	-0.2489
Resultant Length				
ρ_1	0.8811	0.8686	0.7994	0.7607
ρ_2	0.6042	0.5479	0.3482	0.3129
Mean Direction				
μ_0	0.4426	1.2703	1.4450	1.8099
Circular Variance				
v_0	0.1189	0.1314	0.2006	0.2393
Circular Standard Deviation				
σ_0	0.5032 1.0038	0.5308 1.0969	0.6692 1.4526	0.7397 1.5243
Central Trigonometric Moments				
α_1^*	0.8811	0.8686	0.7994	0.7607
α_2^*	0.6007	0.5467	0.3479	0.2830
β_1^*	0	0	0	0
β_2^*	-0.0655	-0.0367	0.0134	0.1337
Skewness γ_1^0	-1.5967	-0.7713	0.1491	1.1416
Kurtosis γ_2^0	-0.1408	-1.3058	-1.5006	-0.9052

VI. PROGRAM LISTINGS

MATLAB tools are employed for developing programs for graphs and computations and code for respective work are presented here.

6.1 Program for graph of characteristic function of Stereographic Semicircular Half Logistic Distribution

```
a=0.2606;%input('enter the value of a =');
th=linspace(0,pi-a,24);
h=pi/24; sig=1.25;%input('enter the value of sigma =');% o.2;
f=(1/sig).*((sec(th/2))).^2.*(1+exp(-(tan(th/2)/sig))).^-2.*exp(-(tan(th/2)/sig));
c=[1 5 1 6 1 5 1]; C1=zeros(1,24); S1=zeros(1,24);
for p=0:23
   yc=cos(p.*th).*f;  ys=sin(p.*th).*f;
   s1=0;  s2=0;
   k=p+1;
   for m=1:4
      for i=1:6
          s1=s1+c(i)*yc(6*(m-1)+i);  s2=s2+c(i)*ys(6*(m-1)+i);
      end
   end
   s1;s2;
re=(3*h/10)*s1;
im=(3*h/10)*s2;
C1(k)=re; S1(k)=im;
end
C1;S1;
phi=C1+1i.*S1
p=0:23;
plot(p,real(phi),'-*g')
```

```
hold on
plot(p,imag(phi),'-*m')
ylabel('\phi')
legend('\sigma=1.25, real(\phi)','imag(\phi)')
title('Characteristic Function of Stereographic Semicircular Half Logistic Distribution')
xlabel('p')
```

6.2 Program for Population Characteristics of Stereographic Semicircular Half Logistic Distribution

```
a=0.187;%input('enter the value of a =');
th=linspace(0+a,pi-a,24);
h=pi/24; sig=1.25 %input('enter the value of sigma =');
f=(1/sig).*((sec(th/2))).^2.*(1+exp(-(tan(th/2)/sig))).^-2.*exp(-(tan(th/2)/sig));
c=[1 5 1 6 1 5 1]; C1=zeros(1,24); S1=zeros(1,24);
for p=0:23
    yc=cos(p.*th).*f;  ys=sin(p.*th).*f;
    s1=0; s2=0;
    k=p+1;
    for m=1:4
        for i=1:6
            s1=s1+c(i)*yc(6*(m-1)+i);  s2=s2+c(i)*ys(6*(m-1)+i);
        end
    end
    s1; s2;
re=(3*h/10)*s1; im=(3*h/10)*s2;
C1(k)=re; S1(k)=im;
end
C1; S1;
phi=C1+1i.*S1;
phi(1)
alpha=[real(phi(2)),real(phi(3))]
beta=[imag(phi(2)),imag(phi(3))]
[mu,rho1,v0,sig0,gamma1,gamma2]=circpropnew(alpha,beta)
```

CIRCPROPNEW

```
function[mu,rho1,v0,sig0,gamma1,gamma2]=circpropnew(alpha,beta)
rho=sqrt((beta(1)).^2+(alpha(1)).^2);
rho;
for i=1:2
   if beta(i)>=0 && alpha(i)>0
        mu(i)=atan(beta(i)/alpha(i));
   end
   if alpha(i)<0
        mu(i)=atan(beta(i)/alpha(i))+pi;
   end
   if beta(i)<0 && alpha(i)>=0
        mu(i)=atan(beta(i)/alpha(i))+2*pi;
   end
   if beta(i)>0 && alpha(i)==0
        mu(i)=pi/2;
   end
end
fprintf('Mean Direction =');
   mu
fprintf('Resultant Length =');
rho1=sqrt((beta).^2+(alpha).^2)
p=1:2;
fprintf('Central Trignometric Moments alpha1 & alpha2: \n')
    p=1:2;
alphag(p)=rho1.*cos(mu-p*mu(1))
fprintf('Central Trignometric Moments beta1 & beta2: \n')
betag(p)=rho1.*sin(mu-p*mu(1))
fprintf('circular variance =');
v0=1-rho1(1) %circular variance;
fprintf('circular standard deviation =');
sig0=sqrt(abs(log(1./((rho1).^2))))% circular standard deviation;
fprintf('skewness =');
gamma1=(betag(2))./(v0.^(3/2))% skewness;
fprintf('kurtosis =');
gamma2=(alphag(2)-((1-v0).^4))./(v0.^2) %kurtosis;
```

VII. CONCLUSIONS

The application of Meijer's G- function in Circular Statistics is employed in the derivation and evaluation of characteristic function and trigonometric moments of Stereographic Semicircular Half Logistic distribution. Using first two trigonometric moments, population characteristics of Stereographic Semicircular Half Logistic distribution are also evaluated.

VIII. REFERENCES

[1] Gradshteyn and Ryzhik, Table of Integrals, series and products, 2007, 7th edition, Academic Press.

[2] Kabe D.G. Some applications of Meijer-G functions to distribution problems in statistics, 1958, *Biometrika*, Volume 45, Issue 3-4, Pages 578–580, https://doi.org/10.1093/biomet/45.3-4.578

[3] Kilbas, Anatoly A., Saxena, Ram K., Saigo, Megumi and Trujillo, Juan J. "The generalized hypergeometric function as the Meijer *G*-function", 2016, *Analysis*, vol. 36, no. 1, pp. 14. https://doi.org/10.1515/anly-2015-5001

[4] Lukacs, E. **Characteristic Functions,** 1970, 2nd ed. Griffin, London.

[5] Mardia, K.V. and Jupp, P.E. (2000). Directional Statistics, *John Wiley, Chichester.*

[6] Phani, Y., Girija, S.V.S. and Dattatreya Rao, A.V., Circular Model Induced by Inverse Stereographic Projection On Extreme-Value Distribution, 2012, IRACST – Engineering Science and Technology: An International Journal (ESTIJ), ISSN: 2250-3498, Vol.2, No. 5, pp. 881 – 888

[7] Pishkoo, M. Darus, Some Applications of Meijer G-Functions as Solutionsof Differential Equations in Physical Models, 2013, Zh. Mat. Fiz. Anal. Geom., Volume 9, Number 3, 379–391

[8] Qureshi, Mohammad Idris and Dar, Showkat Ahmad, "Generalizations and applications of Srinivasa Ramanujan's integral associated with infinite Fourier sine transforms in terms of Meijer's *G*-function" 2021, *Analysis*, vol. 41, no. 3, pp. 145-153. https://doi.org/10.1515/anly-2018-0067

[9] Sreekanth Y, Y. Phani, S.V.S.Girija and A.V. Dattatreya Rao , Stereographic *l* – Axial Half Logistic Distribution, 2018, International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 12, pp. 10627-10634.