Pairwise Exchange Method of Generating A Random Permutation

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Abstract:

This paper deals with a new method of forming a partial random permutation of integers (1,2,…,n) and presents an interesting result. The number of possible permutations using this procedure is less than $n!$ and all the resulting permutations are equiprobable.

Key word: Partial random permutation, pairwise exchange.

1. Introduction

Permutation is a well known concept. A permutation of n elements is a rearrangement of the elements of an ordered list into a one to one correspondence with itself. Permutations and Combinations have always played a significant role in many aspects of Discrete Mathematics and Statistics. The study of random variables associated with randomly constructed permutations continues to attract the attention of many researchers worldwide.

In this article we propose a new method of forming a partial random permutation labeled "Pair wise Exchange Method". The number of possible permutations using this procedure is less than $n!$. Further, an interesting result pertaining to this method is presented.

2. Pair wise Exchange Method of generating a random permutation

For any positive integer n, consider a set of integers $(1,2,3,\dots,j,\dots,2n)$.We now form a new permutation of these integers in the following manner. We denote the resulting permutation by $\pi = (\pi_1,\pi_2,\dots,\pi_{2n})$

At the first stage, randomly choose any two integers say i and j from the above set and exchange their positions with each other.

At the second stage, another pair of integers (i,j), not already chosen at the previous stage, are similarly chosen from the remaining integers and exchanged with one other .

This process is continued till the last two remaining integers are exchanged at the n^{th} stage. When the process terminates, we obtain a new permutation of the set of integers $(1,2,3,\dots,j,\dots,2n)$.

In the above method, without loss of generality we can choose any pair (i,j) with $i<j$. It may be mentioned that the pair wise exchange of integers at each stage will also result in a new permutation. But the resulting permutation of interest will be the one obtained at the n^{th} stage after all pairwise exchanges have taken place.

Illustration 1: Let $\boldsymbol{n=2}$. We consider a set of integers $(\mathbf{1,2,3,4})$.

Stage1: Choose any two integers (i,j) and exchange them. Suppose the pair $(1,2)$ is chosen and their positions exchanged.

Stage 2: The remaining two integers $(3, 4)$ are next chosen and their position exchanged.

This can be shown stepwise as follows.

Stage	Integers chosen $(i.j)$	Arrangement after pair wise exchange
1	(1,2)	(2 1 3 4)
2	(3,4)	(2 1 4 3)

Thus the resulting new permutation is (2 1 4 3) .

For the same set (1 2 3 4) , suppose we had chosen a different pair of integers at the first stage.

Stage	Integers chosen (i,j)	Arrangement after pair wise exchange
1	(2,3)	(1 3 2 4)
2	(1,4)	(4 3 2 1)

Here the resulting new permutation is (4 3 2 1) .

If however, the pairs (1,3) and (2,4) are chosen at the first and second stages respectively and exchanged, the resulting new permutation will be (3 4 1 2) .

We note that the numbers of possible pairs that can be chosen at the first stage are $\binom{4}{2}$. However, each permutation will be repeated twice resulting in only three distinct permutations. We give below the pairs chosen and the resulting permutation.

Pairs selected at stage 1	Pairs selected at stage 2	Resulting permutation
(1, 2)	(3, 4)	(2 1 4 3)
(1, 3)	(2, 4)	(3 4 1 2)
(1, 4)	(2, 3)	(4 3 2 1)
(2, 3)	(1, 4)	(4 3 2 1)
(2, 4)	(1, 3)	(3 4 2 1)
(3, 4)	(1, 2)	(2 1 4 3)

Thus for $n = 2$, the number of distinct permutations of the integers (1 2 3 4) using the above method are (2 1 4 3) , (4 3 2 1) and (3 4 1 2). Since with each chosen pair, we can choose any pair out of the remaining four integers in 3 ways, therefore number of distinct permutations formed is three.

3. Combinatorial Aspects

Theorem 1

The total number of distinct possible permutations of the set of integers $(1,2,3, \dots ,2n)$ is $\frac{2n!}{2^n n!}$

(1)

Proof:

We consider the set of integers $(1, 2, 3, \dots , 2n)$ and the possible permutations that can be generated using the pair wise exchange method.

The first pair of any two integers whose positions are to be exchanged can be chosen randomly at the first stage in $\binom{2n}{2}$ ways. Again, out of the remaining $(2n-2)$ integers, any two integers whose positions are to be exchanged can be chosen to form the second pair at the second stage in $\binom{2n-2}{2}$ ways.

Proceeding in this manner, we are left with only the last two integers at the n^{th} stage i.e the last pair gets chosen automatically.

Hence the total number of possible permutations that can be formed using the above method are given by the product

$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{2}{2}$$

which on simplification gives $\frac{2n!}{2^n}$.

However, each set of n pairs of integers chosen in the successive n stages can appear in $n!$ ways in any permutation.

Hence, the total number of possible distinct permutations will equal

$$\frac{2n!}{2^n n!}.$$

which proves Theorem 1.

It may be mentioned that all the permutations obtained using the above method are equiprobable.

In the next section, we present an interesting result.

4. Expected value of the difference between the first and last integers

Theorem 2

Let $\pi = (\pi_1, \pi_2, \pi_3, \dots, \pi_{2n})$ be the resulting permutation obtained using pairwise exchange method. If π_1 and π_{2n} denote the first and last integers of the resulting permutation, then the expected value of their difference equals unity for all values of n .

i.e $\quad E[\pi_1 - \pi_{2n}] = 1$, for any n. (2)

Proof

π_1 denotes the first element of the resulting permutation π . π_1 can be one of the integers $(2,3,\dots,2n)$ as 1 is exchanged with one of the integers $(2,3,\dots,2n)$. The probability that 1 gets exchanged with any integer i , $(i = 2, 3, \dots, 2n)$ is $\frac{1}{2n-1}$.

Therefore ,

$$E[\pi_1] = \sum_{i=2}^{2n} i\,\frac{1}{2n-1}$$

$$= \frac{1}{2n-1}\left[\frac{2n(2n+1)}{2} - 1\right]$$

$$= \frac{4n^2+2n-2}{2(2n-1)} \qquad (3)$$

Similarly , π_{2n} can take one of the values $(1,2,3,\dots,2n-1)$ as the last integer $2n$ can be exchanged with one of the above integers .

Again the probability that $2n$ gets exchanged with any integer i , $(i = 1,2,3,\dots,2n-1)$ is $\frac{1}{2n-1}$.

Therefore,

$$E[\pi_{2n}] = \sum_{i=1}^{2n-1} i\,\frac{1}{2n-1}$$

$$= \frac{2n(2n-1)}{2(2n-1)}$$

$$= n \qquad (4)$$

Therefore, from (3) and (4), we get

$$E[\pi_1 - \pi_{2n}] = E[\pi_1] - E[\pi_{2n}]$$

$$= \frac{4n^2+2n-2}{2(2n-1)} - n$$

$$= 1 \qquad \text{(on simplification)}$$

Thus, $E[\pi_1 - \pi_{2n}] = 1$ for any n .

which proves Theorem (3) .

Concluding Remarks

We have proposed a partial permutation generation method using pairwise exchange of elements. The number of possible permutations using this procedure is less than $n!$. Further, an interesting result pertaining to this method is presented which proves that, if π_1 and π_{2n} denote the first and last integers of the resulting permutation, then the expected value of their difference equals unity for all values of n .

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