IMPORTANT MATHEMATICAL EQUATIONS DERIVED FROM DE BROGLIE'S MATTER WAVE EQUATION

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Matter wave is one of the most important concepts of Quantum Physics. When an object moves with a certain velocity, it has a wave associated with it which is known as the matter wave. The wavelength of the matter waves can be easily determined by using De Broglie's matter wave equation. It states that if an object of mass (m) is moving with a velocity (v), the wavelength of the matter wave associated with that object can be given by; ʎ = h/mv where (h) is the Planck's constant. The wavelength of this matter wave can also be given by; ʎ = h/p, where (p) is the momentum of the moving object. In most of the cases, the momentum of moving objects are large due to which the wavelengths of the matter waves associated with them is also very small. However, matter waves are prominent in cases of moving microscopic objects like atoms and electrons. In this document, I have derived various important mathematical equations from this matter wave equation. Moreover, equations for relativistic wavelength, relation of this relativistic wavelength with total energy of the moving object, relationship between classical and relativistic wavelengths along with several other mathematical equations derived from the main equation has been provided in this document by me.

As per classical physics, mass is always constant. That means the rest mass of an object and the mass of the object when it is travelling with a certain velocity are same. In relativity, the mass of the object changes with velocity and mass is not constant but rather relative. So, as per relativity, the rest mass of an object and the mass of the same object when it is travelling with a certain velocity are not same. So, there is no concept *of relativistic mass in classical physics. In this document, there are 2 cases. One is classical and another one is relativistic case. In the classical case, the concept of relativistic mass is completely ignored. So, even if the object is moving with very high velocity, its mass will be considered same in magnitude as that of the rest mass of the same object. In the relativistic case, the relativistic mass is taken into consideration. So, even if an object moves with very less velocity, its relativistic mass will be only considered. So, if a person relies on the classical formulas to calculate matter wave's wavelength, it will be wrong as per relativity. If a person has derived matter wave's wavelength by classical formulas, he/she can use that result in the derived equations given in this document to obtain the relativistic wavelength of the matter wave. Other mathematical relations and equations based on matter wave are provided in this document in both classical and relativistic cases. In this way, relations between classical and relativistic wavelengths of matter wave are given in this document.*

Important mathematical relations derived from De-Broglie's matter wave equation

$\lambda_{classical} = \frac{h}{m_0 v} = \frac{h}{P_{classical}}$; where m_0 = rest mass of object, $(P_{classical})$ = momentum of object in classical case, v = velocity of the object.

In classical cases,

Let $(K_{classical})$ be the classical kinetic energy.

$\therefore \frac{1}{2} m_0 v^2 = K_{classical}$; $m_0 v = \frac{2K_{classical}}{v}$.

$$\therefore \boxed{\lambda_{classical} = \frac{hv}{2K_{classical}}}$$

Now, $\frac{h}{2\pi} = \hbar$, where $\hbar$ is Dirac's constant.

$$\therefore h = \hbar 2\pi.$$

$$\therefore \lambda_{classical} = \frac{\hbar \cancel{2} \pi v}{\cancel{2} K_{classical}}$$

$$\Rightarrow \boxed{\lambda_{classical} = \frac{\hbar \pi v}{K_{classical}}}$$

$$\therefore \boxed{K_{classical} = \frac{\hbar \pi v}{\lambda_{classical}}}$$

Let (m_0) be the mass of an object (rest mass of the object) which is moving with velocity (v) and is located very close to the surface of earth. Let (ω) be the weight of that object.

$\therefore\ \omega = m_0 g\ ;\quad m_0 = \dfrac{\omega}{g}$.

$$\therefore\ \boxed{\lambda_{classical} = \frac{hg}{\omega v}}$$

$$\therefore\ \boxed{v = \frac{hg}{\omega(\lambda_{classical})}}$$

Now, $\frac{1}{2} m_0 v^2 = K_{classical}\ ;\ \frac{1}{2}(P_{classical})\, v = K_{classical}$

$$\Rightarrow v = \left(\frac{2K_{classical}}{P_{classical}}\right).$$

Now, $\lambda_{classical} = \dfrac{hg}{\omega v}$

$$\Rightarrow \lambda_{classical} = \frac{hg}{\omega\left(\frac{2K_{classical}}{P_{classical}}\right)}$$

$$\Rightarrow \boxed{\lambda_{classical} = \frac{hg(P_{classical})}{2\omega(K_{classical})}}$$

Now, $\lambda_{classical} = \dfrac{\hbar \cancel{2}\pi g\,(P_{classical})}{\cancel{2}\,\omega\,(K_{classical})}$

$$\therefore\ \boxed{\lambda_{classical} = \frac{\hbar\pi g\,(P_{classical})}{\omega\,(K_{classical})}}$$

$$\therefore \boxed{\omega = \frac{\hbar \pi g \left(P_{classical}\right)}{\left(\lambda_{classical}\right)\left(K_{classical}\right)}}$$

Let (b) be the volume of the object and (ρ) be the density of the object (while not moving).

$$\therefore \rho = \left(\frac{m_0}{b}\right); \quad m_0 = \rho b.$$

$$\therefore \boxed{\lambda_{classical} = \frac{h}{\rho b v}}$$

Now, $$v = \left(\frac{2K_{classical}}{P_{classical}}\right).$$

$$\therefore \lambda_{classical} = \frac{\hbar \cancel{2}\pi \left(P_{classical}\right)}{\rho b \left(\cancel{2}K_{classical}\right)}$$

$$\Rightarrow \boxed{\lambda_{classical} = \frac{\hbar \pi \left(P_{classical}\right)}{\rho b \left(K_{classical}\right)}}$$

$$\therefore \boxed{b = \frac{\hbar \pi \left(P_{classical}\right)}{\rho \left(K_{classical}\right)\left(\lambda_{classical}\right)}}$$

Let the radius of the object be (r) if the object is spherical.

$$\therefore \lambda_{classical} = \frac{h}{\left(\frac{4}{3}\pi r^3\right) \rho v}.$$

$$\Rightarrow \lambda_{classical} = \frac{\hbar 2\pi}{(4/3\,\pi r^3)\rho\, v}$$

$$\Rightarrow \boxed{\lambda_{classical} = \frac{3\hbar}{2r^3 \rho\, v}}$$

$$\therefore \boxed{v = \frac{3\hbar}{2r^3 \rho\,(\lambda_{classical})}}$$

Now, $\left(\frac{2K_{classical}}{P_{classical}}\right) = \frac{3\hbar}{2r^3\rho(\lambda_{classical})}$ [for spherical object].

$$\therefore \boxed{\lambda_{classical} = \frac{3\hbar\,(P_{classical})}{4r^3\rho\,(K_{classical})}}$$

$$\therefore r^3 = \frac{3\hbar\,(P_{classical})}{4\rho\,(K_{classical})(\lambda_{classical})}$$

$$\Rightarrow \boxed{r = \left[\frac{3\hbar\,(P_{classical})}{4\rho\,(K_{classical})(\lambda_{classical})}\right]^{1/3}}$$

In relativistic cases,

$$E^2 = m_0^2c^4 + p^2c^2 \;;\; p^2c^2 = E^2 - m_0^2c^4 \;;\; p^2 = \frac{E^2 - m_0^2c^4}{c^2}$$

$$\therefore p = \frac{\sqrt{E^2 - m_0^2c^4}}{c}$$

$$\therefore \quad \boxed{\lambda_{relativistic} = \frac{hc}{[E^2 - m_0^2c^4]^{1/2}}}$$

Now, $\sqrt{E^2 - m_0^2c^4} = \dfrac{hc}{\lambda_{relativistic}}$

$$\Rightarrow E^2 - m_0^2c^4 = \frac{h^2c^2}{(\lambda_{relativistic})^2}$$

$$\Rightarrow E^2 = \frac{h^2c^2 + m_0^2c^4(\lambda_{relativistic})^2}{(\lambda_{relativistic})^2}$$

$$\Rightarrow E = \frac{\left[(hc)^2 + \{m_0c^2(\lambda_{relativistic})\}^2\right]^{1/2}}{(\lambda_{relativistic})}$$

Now, $E_0 = m_0c^2$ is the energy of the object when it was not moving.

$$\therefore \quad \boxed{E = \left[(hc)^2 + E_0^2(\lambda_{relativistic})^2\right]^{1/2} \cdot (\lambda_{relativistic})^{-1}}$$

Now, $(hc)^2 \approx 0 \qquad \therefore \boxed{E \approx E_0}$

Now, $m = \dfrac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$; $\therefore m = \dfrac{m_0c}{\sqrt{c^2 - v^2}}$.

$$\therefore \quad m(c^2 - v^2)^{1/2} = m_0c$$

$\Rightarrow m^2 (c^2 - v^2) = m_0^2 c^2$

$\Rightarrow m_0^2 c^2 = m^2 c^2 - m^2 v^2$

Now, $\lambda_{relativistic} = \dfrac{hc}{\sqrt{E^2 - m_0^2 c^2 \cdot c^2}}$

$\Rightarrow \lambda_{relativistic} = \dfrac{hc}{\left[E^2 - (m^2c^2 - m^2v^2)c^2\right]^{1/2}}$

$\Rightarrow \lambda_{relativistic} = \dfrac{hc}{\left[E^2 - m^2c^4 + m^2v^2c^2\right]^{1/2}}$

$\therefore \boxed{\lambda_{relativistic} = \dfrac{hc}{\left[E^2 - (mc^2)^2 + (P_{relativistic})^2 c^2\right]^{1/2}}}$

Here, $P_{relativistic} = mv =$ Relativistic momentum.

Now, $E^2 - (mc^2)^2 + m^2v^2c^2 = \dfrac{h^2c^2}{(\lambda_{relativistic})^2}$

Let $(\lambda_r) = (\lambda_{relativistic})$.

$\therefore E^2 = \dfrac{h^2c^2}{(\lambda_r)^2} + (mc^2)^2 - (m^2v^2c^2)$.

As $(h^2c^2) \approx 0$,

$E^2 = (mc^2)^2 - (mvc)^2$

$$\therefore \boxed{E = \left[(mc^2)^2 - (c\,P_{relativistic})^2\right]^{1/2}}$$

Now, $E^2 = \dfrac{h^2c^2}{\lambda_r^2} + (mc^2)^2 - m^2v^2c^2$

$$\Rightarrow E^2 = \frac{(hc)^2 + (mc^2\lambda_r)^2 - (mvc\lambda_r)^2}{\lambda_r^2}$$

$$\therefore \boxed{E = \left[(hc)^2 + (mc^2\lambda_r)^2 - \{\lambda_r c\,(P_{relativistic})\}^2\right]^{1/2} \cdot (\lambda_r)^{-1}}$$

Now, $E^2\lambda_r^2 = (hc)^2 + (mc^2\lambda_r)^2 - (mvc\lambda_r)^2$

$\Rightarrow E^2\lambda_r^2 = (mc^2\lambda_r)^2 - (mvc\lambda_r)^2 \quad [\because (hc)^2 \approx 0]$

$\Rightarrow E^2\cancel{\lambda_r^2} = (m^2c^4 - m^2v^2c^2)\cancel{\lambda_r^2}$

$\Rightarrow m^2(c^4 - v^2c^2) = E^2$

$\Rightarrow m^2 = \dfrac{E^2}{c^4 - v^2c^2}$

$$\Rightarrow \boxed{m = \frac{E}{\sqrt{(c^2+vc)(c^2-vc)}}}$$

When $v \ll c$, $\boxed{m = \dfrac{E}{c^2}}$

In this case,

$\dfrac{E}{c^2} = \dfrac{m_0 c}{\sqrt{c^2 - v^2}}$

$\Rightarrow \dfrac{E}{c^2} = m_0$; $\boxed{E = m_0c^2}$

Now, $m = \dfrac{E}{[(c^2+vc)(c^2-vc)]^{1/2}}$

$\Rightarrow m = \dfrac{E}{[c(c+v)\cdot c(c-v)]^{1/2}} = \dfrac{E}{\sqrt{c^2(c^2-v^2)}}$

$\therefore m = \dfrac{E}{c\sqrt{(c^2-v^2)}}$

$\therefore \dfrac{E}{c\sqrt{c^2-v^2}} = \dfrac{m_0 c}{\sqrt{c^2-v^2}}$

$\Rightarrow E = m_0 c^2$

Now, $mv = \dfrac{Ev}{c\sqrt{c^2-v^2}}$

$$\therefore \boxed{\lambda_r = \frac{hc\sqrt{c^2-v^2}}{Ev}}$$

If we are precise,

$$E = \left[(hc)^2 + (mc^2\lambda_r)^2 - (mvc\lambda_r)^2\right]^{1/2}\cdot(\lambda_r)^{-1}$$

$$\Rightarrow E^2\lambda_r^2 = (hc)^2 + (mc^2\lambda_r)^2 - (mvc\lambda_r)^2$$

$$\Rightarrow E^2\lambda_r^2 - h^2c^2 = m^2c^4\lambda_r^2 - m^2v^2c^2\lambda_r^2$$

$$\Rightarrow E^2\lambda_r^2 - h^2c^2 = m^2\left(c^4\lambda_r^2 - v^2c^2\lambda_r^2\right).$$

$$\Rightarrow m^2 = \frac{E^2\lambda_r^2 - h^2c^2}{c^4\lambda_r^2 - v^2c^2\lambda_r^2}$$

$$\Rightarrow m^2 = \frac{(E\lambda_r)^2 - (hc)^2}{c^2\lambda_r^2(c^2 - v^2)}$$

$$\Rightarrow \quad m = \frac{\left[(E\lambda_r)^2 - (hc)^2\right]^{1/2}}{(c^2\lambda_r^2)^{1/2}\,(c^2-v^2)^{1/2}}$$

$$\therefore \quad \boxed{m = \frac{\left[(E\lambda_r)^2 - (hc)^2\right]^{1/2}}{(c\lambda_r)\,(c^2-v^2)^{1/2}}}$$

$$\therefore \quad \frac{\left[(E\lambda_r)^2 - (hc)^2\right]^{1/2}}{(c\lambda_r)\,\cancel{(c^2-v^2)^{1/2}}} = \frac{m_0 c}{\cancel{(c^2-v^2)^{1/2}}}$$

$$\Rightarrow \quad \boxed{m_0 = \frac{\left[(E\lambda_r)^2 - (hc)^2\right]^{1/2}}{\lambda_r c^2}}$$

Now, $\lambda_r = \dfrac{\left[(E\lambda_r)^2 - (hc)^2\right]^{1/2}}{m_0 c^2}$

$$\therefore \quad \boxed{\lambda_r = \left[(E\lambda_r)^2 - (hc)^2\right]^{1/2} . (E_0)^{-1}}$$

Now, $\lambda_{relativistic} = \dfrac{h}{\left(\dfrac{m_0 c}{\sqrt{c^2-v^2}}\right) v}$

$$\Rightarrow \quad \lambda_{relativistic} = \frac{h\sqrt{c^2-v^2}}{m_0 c v}$$

$$\therefore \quad \boxed{\lambda_r = \frac{h\sqrt{c^2-v^2}}{(P_{classical})\, c}}$$

Let (r) and (ρ) be the radius and density of a spherical object (at rest).

$$\therefore \quad \lambda_r = \frac{h 2\cancel{\pi}\sqrt{c^2-v^2}}{\frac{4}{3}\cancel{\pi} r^3 \rho\, c v}$$

$$\Rightarrow \boxed{\lambda_r = \frac{3\hbar\sqrt{c^2-v^2}}{2r^3\rho c v}}$$

Now, $\frac{1}{2} m_0 v^2 = K_{classical}$; $\frac{1}{2}(m_0 v)\, v = (K_{classical})$

$$\therefore (m_0 v) = \left(\frac{2K_{classical}}{v}\right).$$

$$\therefore \lambda_r = \frac{h\sqrt{c^2-v^2}}{\left(\frac{2K_{classical}}{v}\right) c}$$

$$\Rightarrow \boxed{\lambda_r = \frac{h v\sqrt{c^2-v^2}}{2(K_{classical})\, c}}$$

Now, $\lambda_r = \frac{\hbar \cancel{2}\pi\, v\sqrt{c^2-v^2}}{\cancel{2}c(K_{classical})}$

$$\Rightarrow \boxed{\lambda_r = \frac{\hbar\pi v\sqrt{c^2-v^2}}{c(K_{classical})}}$$

Now, $v = \frac{(2K_{classical})}{(P_{classical})}$

For spherical object,

$$\lambda_r = \frac{3\hbar\sqrt{c^2-v^2}}{2r^3\rho c\left(\frac{2K_{classical}}{P_{classical}}\right)}$$

$$\therefore \boxed{\lambda_r = \frac{3\hbar\sqrt{c^2-v^2}\,(P_{classical})}{4r^3\rho c\,(K_{classical})}}$$

Now, $\lambda_{relativistic} = \frac{h\sqrt{c^2-v^2}}{m_0 v c}$ and $\lambda_{classical} = \frac{h}{m_0 v}$

Now, $\frac{\lambda_{relativistic}}{\lambda_{classical}} = \frac{\cancel{h}\sqrt{c^2-v^2}}{\cancel{m_0 v}c} \times \frac{\cancel{m_0 v}}{\cancel{h}}$

$$\Rightarrow \boxed{\frac{\lambda_{relativistic}}{\lambda_{classical}} = \frac{\sqrt{c^2-v^2}}{c}}$$

$$\therefore \boxed{(\lambda_{relativistic}) = (\lambda_{classical})\left(\frac{\sqrt{c^2-v^2}}{c}\right)}$$

Now, $m = \frac{m_0 c}{\sqrt{c^2-v^2}}$; $\frac{\sqrt{c^2-v^2}}{c} = \left(\frac{m_0}{m}\right)$.

$$\therefore \boxed{\lambda_r = (\lambda_{classical})\left(\frac{m_0}{m}\right)}$$

$$\therefore \boxed{m = \frac{(\lambda_{classical})(m_0)}{(\lambda_{relativistic})}}$$

Let, $(\lambda_c) = (\lambda_{classical})$.

Now, $\frac{(\lambda_c)\cancel{(m_0)}}{(\lambda_r)} = \frac{\cancel{m_0}c}{\sqrt{c^2-v^2}}$

$$\Rightarrow \sqrt{c^2-v^2} = \frac{(\lambda_r)c}{(\lambda_c)}$$

$$\Rightarrow c^2-v^2 = \frac{(\lambda_r)^2 c^2}{(\lambda_c)^2} \; ; \; v^2 = c^2 - \frac{(\lambda_r)^2 c^2}{(\lambda_c)^2}$$

$$\Rightarrow v^2 = c^2 \left\{1 - \frac{(\lambda_r)^2}{(\lambda_c)^2}\right\}.$$

$$\Rightarrow v^2 = c^2 \left\{\frac{(\lambda_c)^2 - (\lambda_r)^2}{(\lambda_c)^2}\right\}.$$

$$\Rightarrow \boxed{v = \left(\frac{c}{\lambda_c}\right)\left[(\lambda_c)^2 - (\lambda_r)^2\right]^{1/2}}$$

Now, $v = \left(\dfrac{2K_{classical}}{P_{classical}}\right)$.

$$\therefore \boxed{\lambda_r = \frac{h\sqrt{c^2-v^2}\,(P_{classical})}{m_0 c\,(2K_{classical})}}$$

$$\therefore \boxed{\lambda_r = \frac{\hbar\pi\sqrt{c^2-v^2}\,(P_{classical})}{m_0 c\,(K_{classical})}}$$

Now, $m = \dfrac{m_0 c}{\sqrt{c^2-v^2}}$.

$$\therefore m_0 c = m\left(\sqrt{c^2-v^2}\right)$$

$$\therefore \boxed{\lambda_r = \frac{\hbar\pi\,(P_{classical})}{(m)(K_{classical})}}$$

Now, $E_0 = m_0 c^2$.

$$\therefore \left(\frac{E_0}{c}\right) = (m_0 c).$$

$$\therefore \boxed{\lambda_r = \frac{\hbar \pi c \sqrt{c^2-v^2}\,(P_{classical})}{(E_0)\,(K_{classical})}}$$

$$\therefore \boxed{E_0 = \frac{\hbar \pi c \sqrt{c^2-v^2}\,(P_{classical})}{(\lambda_r)\,(K_{classical})}}$$

For spherical object,

$$\lambda_r = \frac{3\hbar \sqrt{c^2-v^2}\,(P_{classical})}{4r^3\rho c\,(K_{classical})}$$

Now, $m = \dfrac{m_0 c}{\sqrt{c^2-v^2}}$; $m\sqrt{c^2-v^2} = m_0 c$

$$\Rightarrow \frac{\sqrt{c^2-v^2}}{c} = \left(\frac{m_0}{m}\right).$$

$$\therefore \boxed{\lambda_r = \frac{3\hbar\, m_0\,(P_{classical})}{4r^3\rho\, m\,(K_{classical})}}$$

Now, let (x) and (y) be $(P_{classical})$ and $(K_{classical})$ respectively

$$\therefore \frac{hc}{\sqrt{E^2-m_0^2c^4}} = \frac{3\hbar\, m_0 x}{4r^3\rho m\, y}$$

$$\Rightarrow \frac{\cancel{\hbar}2\pi c}{\sqrt{E^2-m_0^2c^4}} = \frac{3\cancel{\hbar}\, m_0 x}{4r^3\rho m y}.$$

$$\Rightarrow \frac{4\pi^2c^2}{E^2-m_0^2c^4} = \frac{9m_0^2x^2}{16r^6\rho^2m^2y^2}$$

$$\Rightarrow E^2-m_0^2c^4 = \frac{(4\pi^2c^2)(16r^6\rho^2m^2y^2)}{9m_0^2x^2}$$

$$\Rightarrow E^2 = \frac{(4\pi^2c^2)(16 r^6 \rho^2 m^2 y^2)}{9 m_0^2 x^2} + m_0^2 c^4$$

$$\Rightarrow E^2 = \frac{(4\pi^2c^2)(16 r^6 \rho^2 m^2 y^2)}{9 m_0^2 x^2} + E_0^2$$

$$\Rightarrow E^2 = \frac{(4\pi^2c^2)(16 r^6 \rho^2 m^2 y^2) + 9 m_0^2 x^2 E_0^2}{9 m_0^2 x^2}$$

$$\Rightarrow E^2 = \frac{(4\pi^2c^2)(4 r^3 \rho m y)^2 + 9 m_0^2 x^2 E_0^2}{(3 m_0 x)^2}$$

$$\Rightarrow E^2 = \frac{(4\pi^2c^2)(4 r^3 \rho m \times \frac{1}{2} \times x \times v)^2 + (3 m_0 x E_0)^2}{(3 m_0 x)^2}$$

$$\Rightarrow E^2 = \frac{(2\pi c)^2 (2 r^3 \rho m x v)^2 + (3 m_0 E_0)^2 (x)^2}{(3 m_0)^2 x^2}$$

$$\Rightarrow E^2 = \frac{(2\pi c)^2 (2 r^3 \rho m v)^2 + (3 m_0 E_0)^2}{(3 m_0)^2}$$

Let, (a) = relativistic momentum = (mv).

$$\therefore E^2 = \frac{(2\pi c)^2 (2 r^3 \rho a)^2 + (3 m_0 E_0)^2}{(3 m_0)^2}$$

$$\therefore \boxed{E = \left[(2\pi c)^2 (2 r^3 \rho a)^2 + (3 m_0 E_0)^2\right]^{1/2} \cdot (3 m_0)^{-1}}$$

(Only applicable for spherical object)