On 3-dimensional f-Kenmotsu Manifolds with semi-symmetric metric connection

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Abstract The main purpose of this paper is to exploration about f-Kenmotsu manifolds with semi-symmetric metric connections . Some necessary and sufficient conditions for such manifolds to be η-parallel are studied. Ricci tensor,Projective curvature tensor and Concircular curvature tensor have been studied on this manifold . Also we have constructed an example of a of f-kenmotsu manifolds to justify our results.

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1 Introduction

In 1958, M.M.Boothby and R.C.Wong [8] introduce the concept of odd dimensional manifold with contact and almost contact structure . K.Kenmotsu[6] in 1972 studied a class of almost contact metric manifolds and called them as a Kenmotsu Manifold. He also proved that if a Kenmotsu manifold satisfies the condition of $R(X,Y).R = 0$ then the manifold became of negative curvature -1 , where R is the Riemanian curvature tensor of type $(1,3)$. In 1924, Friedman and Schouten[1] introduced the notion of semi-symmetric metric connections on a differentiable manifold and the notion of semi-symmetric metric connection was defined and studied by S.Golab[10] . The notion of quarter symmetric metric connection is a generelization of semi-symmetric metric connection. A linear connection is said to be a semi-symmetric metric connection if its torsion tensor T is of the form,

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where η is a 1-form.

In 1970, K.Yano[7] considered a semi-symmetric metric connection and studied some of its properties.

2 Preliminaries

Let M be a 3- dimensional differentiable manifold with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a (1,1)-tensor field ,ξ is a vector field,η is a 1 -form and g is a Riemannian metric on the manifold , satisfying the relations

$$\phi^2 X = -X + \eta(X) \otimes \xi, \tag{2.1}$$
$$\eta(\xi) = 1, \tag{2.2}$$
$$\eta(X) = g(X,\xi), \tag{2.3}$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$
$$g(X,\phi Y) = -g(\phi X, Y), \tag{2.5}$$
$$\phi\xi = 0, \tag{2.6}$$
$$\eta \circ \phi = 0, \tag{2.7}$$

for any vector fields X,Y on the manifold M. The manifold M is called an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies the relation [5]

$$(\nabla_X \phi)Y = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.8}$$

where ∇ is the Levi-Civita connection of the f-Kenmotsu manifolds and f is a c^{∞}- function on the manifold . If f=β=constant $\neq 0$,then the manifold is β-Kenmotsu manifold [13] and if $f = 0$, then the manifold reduces to cosympletic manifold [13]. Moreover f-Kenmotsu manifold is called regular if $f^2 + f' \neq 0$,where $f' = \xi f$. From equation (2.2) ,we get

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{2.9}$$

In a 3- dimensional f-Kenmotsu manifold, we have [2]

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y). \tag{2.10}$$

$$R(X,Y)\xi = -(f^2 + f')(\eta(Y)X - \eta(X)Y), \tag{2.11}$$

$$R(\xi,Y)Z = -(f^2 + f')(g(Y,Z)\xi - \eta(X)Y), \tag{2.12}$$

$$S(X,\xi) = -2(f^2 + f')\eta(X), \tag{2.13}$$

$$S(\xi,\xi) = -2(f^2 + f'), \tag{2.14}$$

$$Q\xi = -2(f^2 + f')\xi \quad , \tag{2.15}$$

where R, S and Q denote the Riemanian curvature tensor, Ricci tensor and Ricci operator respectively. As a consequence of equation (2.9) we also have,

$$(\nabla_X \eta)(Y) = fg(\phi X, \phi Y), \tag{2.16}$$

Also we have,

$$S(\phi X, \phi Y) = S(X,Y) + 2(f^2 + f')\eta(X)\eta(Y), \tag{2.17}$$

for all vector fields X,Y. In a 3-dimensional f-Kenmotsu manifold we also have [10]

$$R(X,Y)Z = (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z - (\frac{r}{2} + 3f^2 + 3f')(\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z), \tag{2.18}$$

and

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.19}$$

where r is the scalar curvature and $f' = \xi f$.

3 f-Kenmotsu manifolds with semi-symmetric metric connection

Let M be a 3-dimensional f- Kenmotsu manifold and ∇ denotes the Levi-Civita connection on it . A linear connection $\tilde{\nabla}$ on M is said to be a semi-symmetric if the torsion tensor $\tilde{T}$ of type (1,2) defined by,

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \tag{3.1}$$

satisfies,

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{3.2}$$

for all vector fields X and Y on M . If moreover , the semi-symmetric connection $\widetilde{\nabla}$ holds the relation $\widetilde{\nabla}_g = 0$ is called semi-symmetric metric connection. A semi-symmetric connection $\widetilde{\nabla}$ is said to be non-metric if $\widetilde{\nabla}_g \neq 0$. A relation between a semi-symmetric metric and Levi-Civita connections is given as ,

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{3.3}$$

for all vector fields $X, Y \in \chi(M)$ ([22]). With the help of equations (2.1),(2.2), (2.3),(2.4),(2.5) we can easily observe that

$$(\widetilde{\nabla}_x \eta)(Y) = (\nabla_x \eta)Y - \eta(X)\eta(Y) + g(X,Y), \tag{3.4}$$

and

$$\widetilde{\nabla}_X \phi(Y) = -g(X,\phi Y)\xi - 2\eta(Y)\phi X, \tag{3.5}$$

If R and $\tilde{R}$ denotes the curvature tensors with respect to the Levi-Civita and semi-symmetric metric connections of the manifold M respectively,then

$$\tilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}[X,Y], \tag{3.6}$$

for all $X, Y \in \chi(M)$. In consequence of equations $(2.1), (2.2), (2.3), (2.4)$,we have

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + 2fg(X,Z)Y - 2fg(Y,Z)X \\ &+ f\eta(Z)(X\eta(Y) - Y\eta(X)) + f'(\eta(X)g(Y,Z) \\ &- \eta(Y)g(X,Z)) + \eta(Z)(Y\eta(X) - X\eta(Y)) \\ &- Xg(Y,Z) + Yg(X,Z) + (\eta(X)g(Y,Z) - \eta(Y)g(X,Z))\xi, \end{aligned} \tag{3.7}$$

for all $X, Y, Z \in \chi(M)$, we can write ,

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z - (f+1)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \\ &- (f+1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ &+ (2f+1)[g(X,Z)Y - g(Y,Z)X], \end{aligned} \tag{3.8}$$

now contracting equation(3.8) with respect to the vector field X ,we get

$$\tilde{S}(Y,Z) = S(Y,Z) + (f+1)\eta(Y)\eta(Z) - (3f+1)g(Y,Z), \tag{3.9}$$

which is similar to

$$\tilde{Q}Y = QY + (f+1)\eta(Y)\xi - (3f+1)Y, \tag{3.10}$$

where $\tilde{Q}$ and Q denotes the Ricci operators corresponding to the respective connections $\widetilde{\nabla}$ *and* ∇ and now we have to defined $\tilde{S}(Y,Z) = g(\tilde{Q}Y,Z)$ and $S(Y,Z) = g(QY,Z)$. Let $e_i, i = 1,2,3,\ldots,n$ be an orthonormal basis of the tangent space at every point of manifold M. Now considering $Y = Z = e_i$ in equation (3.10) and taking summation over i,1≤i≤n, we have

$$\tilde{r} = r - 8f - 2, \tag{3.11}$$

where

$$\tilde{r} = \sum_{i=1}^{n} \tilde{S}(e_i, e_i), \tag{3.12}$$

which represents the scalar curvatures with respect to the connections $\widetilde{\nabla}$ and ∇ respectively.
A f-Kenmotsu manifold is said to be η-Einstein Manifold if its Ricci tensor S takes the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{3.13}$$

for arbitrary vector fields X and Y, where a and b are smooth functions on manifold (M,g)[5]. If b=0, then η-Einstein manifold becomes Einstein manifold.

Theorem 3.1 *Let M be a 3- dimensional f-Kenmotsu manifold equipped with a semi-symmetric metric connection $\widetilde{\nabla}$. Then the Ricci tensor $\tilde{S}$ on M is η-einstein with respect to semi-symmetric metric connection $\widetilde{M}$.*

Proof. Let M be a 3- dimensional f-Kenmotsu manifold with a semi-symmetric metric connection $\widetilde{\nabla}$. We know that Ricci tensor for f-Kenmotsu manifold is given as

$$\tilde{S}(Y,Z) = S(Y,Z) + (f+1)\eta(Y)\eta(Z) - (3f+1)g(Y,Z), \tag{3.14}$$

if we consider $S(X,Y) = 0$,then we get Ricci tensor is of the form

$$\tilde{S}(Y,Z) = (f+1)\eta(Y)\eta(Z) - (3f+1)g(Y,Z). \tag{3.15}$$

This shows that Ricci tensor $\tilde{S}$ is of the form of η-Einstien manifold.

4 η-parallel Ricci tensor with semi-symmetric metric connection

Many authors studied the properties of η-parallel Ricci tensor and proved several results on it [8].In this section we have to discussed about the various geometrical properties of η-parallel Ricci tensor with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. A Ricci tensor $\tilde{S}$ of an n-dimensional Kenmotsu manifold M endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ is said to be η-parallel if it satisfies the following relation

$$(\widetilde{\nabla}_X\tilde{S})(\phi Y, \phi Z) = 0, \tag{4.1}$$

for arbitrary vector fields X , Y and Z.

Theorem 4.1 *Let M be a 3- dimensional f-Kenmotsu manifold equipped with a semi-symmetric metric connection $\widetilde{\nabla}$. Then the Ricci tensor S is η-parallel with respect to semi-symmetric metric connection $\widetilde{\nabla}$ on the manifold M.*

Proof. Let M be a 3- dimensional f-Kenmotsu manifold with a semi-symmetric metric connection $\widetilde{\nabla}$. We have

$$(\nabla_X\phi)Y = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{4.2}$$

$$\tilde{S}(X,Y) = (\tfrac{r}{2} + f^2 + f')g(X,Y) - (\tfrac{r}{2} + 3f^2 + 3f')(\eta(X)\eta(Y)), \tag{4.3}$$

now putting X=ϕX and Y=ϕY in (4.3),we have

$$\tilde{S}(\phi X, \phi Y) = (\tfrac{r}{2} + f^2 + f')g(\phi X, \phi Y) - (\tfrac{r}{2} + 3f^2 + 3f')(\eta(\phi X)\eta(\phi Y)), \tag{4.4}$$

as we know that $\eta\phi = 0$ and using the equation which is given below

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{4.5}$$

then we have

$$\tilde{S}(\phi X, \phi Y) = (\tfrac{r}{2} + f^2 + f')[g(X,Y) - \eta(X)\eta(Y)], \tag{4.6}$$

since we know that

$$\eta(X)=g(X,\xi),\eta(Y)=g(Y,\xi) \tag{4.7}$$

so we get

$$\tilde{S}(\phi X,\phi Y)=(\frac{r}{2}+f^2+f')[g(X,Y)-g(X,\xi)g(Y,\xi)], \tag{4.8}$$

now differentiating equation (4.8) covariantly with respect to vector field Z, we get

$$\tilde{\nabla}_Z\tilde{S}(\phi X,\phi Y)=0. \tag{4.9}$$

as $\nabla_X g(Y,Z)=0$

Hence S is η-parallel with respect to semi-symmetric metric connection $\tilde{\nabla}$ on the manifold M.

Theorem 4.2 *If the Ricci tensor $\tilde{S}$ of a 3-dimensiional f-Kenmotsu manifold M equipped with a semi-symmetric metric connection $\tilde{\nabla}$ is η-parallel, then the scalar curvature for the semi-symmetric metric connection $\tilde{\nabla}$ is $-8f'$.*

Proof. Let M be a 3-dimensional f-Kenmotsu manifold with semi-symmteric metric connection $\tilde{\nabla}$. Let $e_i, i=1,2,3,\ldots n$ be an orthonormal basis of the tangent space at any point of the manifold M.
From equation(3.11) we have scalar curvature $\tilde{r}$ of the form ,

$$\tilde{r}=r-8f-2, \tag{4.10}$$

Now differentiate $\tilde{r}$ with respect to vector field X .

$$d\tilde{r}(X)=-8f' \tag{4.11}$$

where $f'=\xi f$.

5 Concircular curvature tensor with semi-symmetric metric connection

It is known that a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) is not tranformed into a geodesic circle by the conformal transformation

$$\bar{g}_{ij}=\psi^2 g_{ij}, \tag{5.1}$$

where g_{ij} denotes the fundamental tensor.K.Yano [7] proved about the conformal transformation ,which is well-defined in eaquation (5.1),satisfying the partial differential equation

$$\psi_{i;j}=\phi g_{ij} \tag{5.2}$$

It is clearly alter a geodesic circle into a geodesic circle.Such kind of transformation is concircular transformation and the geometry deals with such transformation is known as concircular geometry [14] . A tensor field of type (1,3) on Riemannian manifold , which remains invariant under the concircular transformation defined by,

$$C(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}[g(Y,Z)X-g(X,Z)Y], \tag{5.3}$$

where R is the curvature tensor and r denotes the scalar curvature ,is known as concircular curvature tensor [11] .

Let M be a 3- dimensional f-Kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. A concircular curvature tensor $\tilde{C}$ with semi-symmetric metric connection $\tilde{\nabla}$ on the manifold M is a tensor field of type (1,3) defined as,

$$\tilde{C}(X,Y)Z=\tilde{R}(X,Y)Z-\frac{\tilde{r}}{n(n-1)}[g(Y,Z)X-g(X,Z)Y], \tag{5.4}$$

for all vector fields $X,Y,Z \in \chi(M)$.Here $\tilde{R}$ and $\tilde{r}$ are the curvature tensor and scalar curvature of the manifold M with semi symmetric metric connection $\tilde{\nabla}$ respectively.

Let M be a 3-dimensional f- kenmotsu manifold equipped with a semi-symmetric metric connection $\tilde{\nabla}$. A Projective curvature tensor $\tilde{P}$ with semi-symmetric metric connection $\tilde{\nabla}$ on the manifold M is a tensor field of type (1,3) is given as,

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y] \tag{5.5}$$

for all vector fields $X, Y, Z \in \chi(M)$. Here $\tilde{R}$ and $\tilde{S}$ denotes the curvature tensor and Ricci tensor of the manifold M with semi-symmetric metric connection $\tilde{\nabla}$ respectively.

Theorem 5.1 *A 3-dimensional f-Kenmotsu manifold is always ξ-projectively flat.*

Proof. Let M ba a 3-dimensional f-Kenmotsu manifold .The Riemannian curvature tensor $R(X,Y)Z$ and Ricci tensor $S(X,Y)$ in 3-dimensional f-Kenmotsu manifold is defined as,

$$R(X,Y)Z = (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z - (\frac{r}{2} + 3f^2 + 3f')(\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z). \tag{5.6}$$

and

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{5.7}$$

where r is the scalar curvature and $f' = \xi f$.
From equation (5.6) putting $Z = \xi$,we have

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y]. \tag{5.8}$$

In equation(5.7) putting $Y = \xi$, we have

$$S(X,\xi) = -2(f^2 + f')\eta(X). \tag{5.9}$$

using equations (5.8) and (5.9) ,in equation(5.5), we have

$$\tilde{P}(X,Y)\xi = 0. \tag{5.10}$$

which shows that a 3-dimensional f-Kenmotsu manifold is always ξ-projectively flat.

Theorem 5.2 *If the Ricci tensor $\tilde{S}$ of a 3- dimensional f-Kenmotsu manifold M endowed with a semi-symmetric metric connection $\tilde{\nabla}$ is η-parallel. Then the following results on M are equivalent ,*
i)Ricci tensor is cyclic parallel,
ii)$\tilde{S}(Y,Z) = -2(n-1)g(Y,Z)$,
iii)$\tilde{r} = -2n(n-1)$

Proof (i): Let M be a f-Kenmotsu manifold with respect to semi-symmetric metric connection. We have

$$\eta(\xi) = 1, \tag{5.11}$$
$$\phi^2 = -I + \eta \otimes \xi, \tag{5.12}$$
$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{5.13}$$
$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{5.14}$$
$$\tilde{S}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{5.15}$$

from equations (5.11),(5.12),(5.13),(5.14) and (5.15),we get

$$\tilde{S}(\phi Y, \phi Z) = \tilde{S}(Y, Z) + 2(n-1)\eta(Y)\eta(Z), \tag{5.16}$$

now differentiating covarinatly equation (5.16) with respect to X, then we have

$$\widetilde{\nabla}_X \tilde{S}(\phi Y, \phi Z) = \widetilde{\nabla}_X S(Y, Z) - \tilde{S}(\widetilde{\nabla}_X(\phi Y), \phi Z) - \tilde{S}(\phi Y, \widetilde{\nabla}_X(\phi Z)), \tag{5.17}$$

and since we clearly have

$$\widetilde{\nabla}_X \tilde{S}(\phi Y, \phi Z) = \widetilde{\nabla}_X \tilde{S}(Y, Z), \tag{5.18}$$

after some calculation ,we get

$$\begin{aligned}\widetilde{\nabla}_X \tilde{S}(Y, Z) = &-2\eta(Y)\tilde{S}(X, Z) + \eta(Z)\tilde{S}(X, Y) \\ &-4(n-1)\eta(Y)g(X, Z) + \eta(Z)g(X, Y),\end{aligned} \tag{5.19}$$

$$\begin{aligned}\widetilde{\nabla}_Y \tilde{S}(Z, X) = &-2\eta(Z)\tilde{S}(Y, X) + \eta(X)\tilde{S}(Y, Z) \\ &-4(n-1)\eta(Z)g(Y, X) + \eta(X)g(Y, Z),\end{aligned} \tag{5.20}$$

$$\begin{aligned}\widetilde{\nabla}_Z \tilde{S}(X, Y) = &-2\eta(X)\tilde{S}(Z, Y) + \eta(Y)\tilde{S}(Z, X) \\ &-4(n-1)\eta(X)g(Z, Y) + \eta(Y)g(Z, X),\end{aligned} \tag{5.21}$$

now adding equations (5.19),(5.20)and (5.21) then we have,

$$\widetilde{\nabla}_X \tilde{S}(Y, Z) + \widetilde{\nabla}_Y \tilde{S}(Z, X) + \widetilde{\nabla}_Z \tilde{S}(X, Y) = 0. \tag{5.22}$$

i.e Ricci tensor $\tilde{S}$ is cyclic parallel.

Proof (ii): We know that

$$\eta(\xi) = 1, \phi^2 = -I + \eta \otimes \xi, \tag{5.23}$$

we have

$$S(X, \xi) = -(n-1)\eta(X), \tag{5.24}$$

$$\tilde{S}(Y, Z) = S(Y, Z) - (3n-5)g(Y, Z) + 2(n-2)\eta(Y)\eta(Z), \tag{5.25}$$

$$\tilde{S}(Y, \xi) = -2(n-1)\eta(Y), \tag{5.26}$$

we know that

$$\eta(X) = g(X, \xi) \tag{5.27}$$

so we have ,

$$\tilde{S}(Y, \xi) = -2(n-1)g(Y, \xi), \tag{5.28}$$

Proof (iii): Let $e_i, i = 1,2,3, \dots n$ be an orthonormal basis of the tangent space at any point of the f-Kenmotsu manifold M . Putting $Y = Z = e_i$ in equation(5.28) and taking summation over i, 1<i<n ,then we get

$$\tilde{r} = -2n(n-1). \tag{5.29}$$

Theorem 5.3 *In a 3-dimensional f-Kenmotsu manifold M with Levi-civita connection ∇, we have*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \tag{5.30}$$

Proof. Let M be a 3-dimensional f-Kenmotsu manifold with semi-symmetric metric connection ∇. we know that Ricci curvature tensor

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ -\frac{r}{2}(g(Y,Z)X - g(X,Z)Y), \quad (5.31)$$

In similar way we have

$$R(Y,Z)X = g(Z,X)QY - g(Y,X)QZ + S(Z,X)Y - S(Y,X)Z \\ -\frac{r}{2}(g(Z,X)Y - g(Y,X)Z),$$

(5.32)

$$R(Z,X)Y = g(X,Y)QZ - g(Z,Y)QX + S(X,Y)Z - S(Z,Y)X \\ -\frac{r}{2}(g(X,Y)Z - g(Z,Y)X), \quad (5.33)$$

On adding equation (5.31),(5.32),(5.33) ,we have

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\ = g(Y,Z)QX + g(Z,X)QY + g(X,Y)QZ - (X,Z)QY - (Y,X)QZ - g(Z,Y)QX \\ +S(Y,Z)X - S(X,Z)Y + S(Z,X)Y - S(Y,X)Z + S(X,Y)Z - S(Z,Y)X \\ -\frac{r}{2}g(Y,Z)X - g(X,Z)Y + g(Z,X)Y - g(Y,X)Z + g(X,Y)Z - g(Z,Y)X, \quad (5.34)$$

since g is Riemannian metric which is symmetric and Ricci tensor S which is also symmetric i.e

$$g(Z,X) = g(X,Z), \qquad S(X,Y) = S(Y,X)$$

so we have ,

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \quad (5.35)$$

6 Example

We consider a 3-dimensional f-Kenmotsu manifold with semi-symmetric metric connections

$$\widetilde{M} = [x_1, x_2, t \in R^3 : t \neq 0] \quad (6.1)$$

where (x_1, x_2, t) are the standadrd coordinates in R. Let us choose the vector fields

$$e_1 = t^2 \frac{\partial}{\partial x_1}, \quad (6.2)$$

$$e_2 = t^2 \frac{\partial}{\partial x_2}, \quad (6.3)$$

$$e_3 = \frac{\partial}{\partial_t}, \quad (6.4)$$

which are linearly independent at each point of $\widetilde{M}$. Now we define the metric g such that e_1, e_2, e_3 is an orthonormal basis of $\widetilde{M}$ i.e,

$$g(e_i, e_j) = 1 \quad if \quad i = j$$
$$= 0 \quad if \quad i \neq j, \quad where \quad 1 < i, j < 3. \tag{6.5}$$

Now consider a 1-form η which is defined by

$$\eta(X) = g(X, e_3), \qquad X \in \chi(\widetilde{M})$$

We choose $e_3 = \xi$. We define the tensor field ϕ by $\phi(e_1) = -e_2,\ \phi(e_2) = e_1,\ \phi(e_3) = 0.$

The linear property of g and ϕ showing that

$$\eta(e_3) = 1, \qquad \phi^2(X) = -X + \eta(X)e_3, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
(6.6)

for any vector fields X, Y on $\widetilde{M}$. Then $\widetilde{M}(\phi, \xi, \eta, g)$ forms an almost contact manifold with $e_3 = \xi$. Let ∇ be the levi-civita connection with respect to the metric g , then we have

$$[e_3, e_i] = \frac{2}{t} e_i \quad , i = 1,2,3 \qquad and \quad [e_i, e_j] = 0 \tag{6.7}$$

otherwise . Now using Koszul's formula , we get

$$\nabla_{e_1} e_1 = (2/t)e_3, \tag{6.8}$$
$$\nabla_{e_1} e_3 = -(2/t)e_1, \tag{6.9}$$
$$\nabla_{e_2} e_3 = -(2/t)e_2, \tag{6.10}$$
$$\nabla_{e_2} e_2 = (2/t)e_3, \tag{6.11}$$
$$\nabla_{e_3} e_3 = (2/t)e1, \tag{6.12}$$
$$\nabla_{e_i} e_j = 0, otherwise. \tag{6.13}$$

The above relations implies that the manifold satisfies

$$\nabla_X \xi = f(X - \eta(X)\xi) \tag{6.14}$$

for $\xi = e_3$, and $f = -\frac{2}{t}$. Hence we can say that M is an f-Kenmotsu manifold . Again since $f^2 + f' \neq 0$, so that the manifold is callled regular f-Kenmotsu manifold. Let M be a subset of $\widetilde{M}$ and consider the immersion $h: M \to \widetilde{M}$ defined by $h(x_1, x_2, t) = (x_1, x_2, 0,0, t)$. It is easy to prove that $M = (x_1, x_2, t) \in R^3 : t \neq 0$ is a subamnifold $\widetilde{M}$ where (x_1, x_2, t) are the standard coordinates of (e_1, e_2, e_3). now we choose the vector fields such as

$$e_1 = t^2 \frac{\partial}{\partial x_1}, \tag{6.15}$$
$$e_2 = t^2 \frac{\partial}{\partial x_2}, \tag{6.16}$$
$$e_3 = \frac{\partial}{\partial t}. \tag{6.17}$$

We define g_1 such that $[e_1, e_2, e_3\ ]$ is an orthonormal basis of M. that is ,

$$g_1(e_i, e_j) = 1 \qquad if \quad i = j \tag{6.18}$$
$$= 0 \qquad if \quad i \neq j, \tag{6.19}$$

$where \quad i, j$
$= 1,2,3.$

We define a 1-form η_1 and a (1,1)tensor ϕ_1 as,

$$\eta_1 = g_1(X, e_3), \qquad \phi_1(e_1) = -e_2, \qquad \phi_1(e_2) = e_1, \qquad \phi_1(e_3) = 0, \quad (6.20)$$

The linearity property of g_1 and ϕ_1 shows that

$$\eta_1(e_3) = 1, \qquad \phi^2 X = -X + \eta_1(X)e_3, \quad (6.21)$$

$$g_1(\phi_1 X, \phi_1 Y) = g_1(X, Y) - \eta_1(X)\eta_1(Y), \quad (6.22)$$

for any vector fields X ,Y on $M(\phi_1, \xi, \eta_1, g_1)$. It is seen that M is manifold of $\widetilde{M}$ with $e_3 = \xi$. Moreover , let ∇ be the Levi-civita connection with respect to the given metric g_1 [3][4] .
Then we have following results

$$[e_3, e_i] = \frac{2}{t} e_i, i = 1,2 \qquad and \qquad [e_i, e_j] = 0, \qquad otherwise. \quad (6.23)$$

Again using Koszul's formula , we obtain the following

$$\nabla_{e_1} e_1 = (\frac{2}{t} - 1)e_3, \quad (6.24)$$
$$\nabla_{e_1} e_3 = (\frac{-2}{t} + 1)e_1, \quad (6.25)$$
$$\nabla_{e_2} e_3 = -\frac{2}{t} e_3 + e2, \quad (6.26)$$
$$\nabla_{e_2} e_2 = (\frac{2}{t} - 1)e_3, \quad (6.27)$$
$$\nabla_{e_i} e_j = 0, otherwise. \quad (6.28)$$

References

[1] A.Friedman and J.A.Schouten,Uber die geometric derhalbsymmetrischen Ubertragung,Math. Zeitscr.,21(1924),211-223.

[2] A.Shafrudddin and S.I.Husain,Semi-symmetric metric connections in almostcontact manifolds ,Tensor N.S.30(1976),133-139.

[3] A.Yildiz,U.C.De and M.Turan, On 3-diemnsional f-Kenmotsu manifolds and Ricci solitons ,Ukrainian Mathematical J.,65(2013),620-628.

[4] C.Murathan, K.Arslan and E.Ezentas , Ricci generalizespseudo-symmetric immersions ,Differential geometry and its applications ,Matfyzpress,Prague,99-108(2005).

[5] D.Janssens , L.Vanhecke ,Almost contact structure and curvature tensor ,Kodai Math .J.4(1981),1-27.

[6] K.Kenmotsu , A class of almost contact Riemannian manifolds , Tohuku Math.j.24(1972),93-103.

[7] K.Yano, and M.Kon, Structures on manifoldsWorld Scientific publication,1984.

[8] M.M.Boothby and R.C.Wong, On Contact manifolds ,Anna.Math.68(1958),421-450.

[9] M.Kon, Invariant submanifolds of normal contact metric manifolds, Kodai Math.Sem. Repors,25(1973),330-336.

[10] S.Golab, On a semi-symmetric and quarter-symmetric linear connections,Tensor,N.S., 29(1975),249-

254.

[11] S.Mukhopadhyay, A.K.Roy, and B.Barua ,Some properties of a quarter-symmetric metric connection on a Riemannian manifold , Soochow J. of Math.,17(1991),205-211.

[12] T.Demirli,C.Ekici and A.Gorgulu,Ricci solitons in f-Kenmotsu manifolds with the semi-symmetric non-metric connection,New Trends in Mathematical Sci.,4(2016),276-284.

[13] W.Roter, On conformally recurrent Ricci -reccurent manifolds , Colloq Math.,46(1982),45-57.

[14] Z. Olszak and R.Rosca, Normal locally conformal almost cosympletic manifolds ,Publications Mathematicae Debrecen,39(1991),315-323.