An Image Inpainting Model for Grayscale Images Based on Edge Enhancing TV-H^{-1} Equation

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Abstract - Image inpainting is the interpolation of missing or damaged portions of images employing information from the boundary and adjacent areas. Several fourth order Partial Differential Equation (PDE) based models are available in the literature to solve the inpainting problem, e.g., various Curvature Driven Diffusion methods, Cahn Hilliard Equation, TV-H^{-1} etc. This paper presents a new fourth order PDE for image inpainting based on TV-H^{-1} coupled with the edge enhancing structure tensor. The edge enhancing structure tensor has been extensively used by J. Weickert in his formulation of anisotropic diffusion equation. It enhances diffusion in homogenous regions but prohibits diffusion across edges. The results indicate that the proposed method generates far better results than state-of-the-art methods.

Index Terms - Image inpainting, Total Variation, Structure Tensor, Sobolev Dual Space, Cahn–Hilliard equation

I. Introduction

Image Inpainting is filling in missing parts of damaged images based on information extracted from surrounding areas. This problem can be considered an interpolation problem. The image inpainting problem has a wide range of applications, from restoring antique paintings to reducing specular reflections in biomedical images and many others.

Mathematically, image inpainting is the problem of reconstructing the image u from a given damaged image f. The image domain is denoted by Ω, and the damaged domain is denoted by $D \subset \Omega$, i.e., D is a subset of the image domain Ω.

Many mathematical inpainting models have been proposed in the last few decades, e.g., exemplar-based inpainting, stochastic, wavelet, and interpolation methods. But it was the PDE-based models which gained more popularity.

Bertalmio et al. [1] was the pioneer in this domain. They devised a nonlinear PDE model which propagated image information (the Laplacian of the image) in the direction of the sharp isophotes (lines of the same grey values, typically edges) continuously into the interior of the regions to be inpainted. This PDE model is known as transport inpainting model.

Subsequently, more PDE-based models for image inpainting were devised, such as the Total Variation (TV) models [2], [3] proposed by Rudin, Osher and later by Chan and Shen, were second order PDEs. It was found that these image inpainting models could not connect the edges over longer distances or smoothly propagate isophotes into the damaged areas. Another third-order variational approach was devised, which was named Curvature Driven Diffusion (CDD) method [4], [5]. Still, it was found that this method may introduce artefacts in the isophotes along the boundary of the inpainting areas.

All these drawbacks in these models point to the fact that higher-order PDE models are needed for better inpainting performance. Consequently, several fourth order PDE models like the Cahn-Hilliard model and TV-H^{-1} model are proposed for image inpainting [6], [7].

This paper proposes a fourth order PDE model for image inpainting based on TV-H^{-1} equation coupled with Edge Enhancing Structure Tensor. The proposed model is evaluated on several grayscale images. The visual results as well as performance measures show that the proposed model produces better inpainting results in less computational time than the Cahn Hilliard Perona Malik model [8] for grayscale images. The results also suggest that the proposed model is better than many state-of-the-art PDE-based image inpainting models.

II. Literature Survey

Cahn-Hilliard Inpainting Model

The Cahn Hilliard Inpainting model discussed in [6] is applicable only for black and white images, i.e., images with pixel values of either 0 or 1.

Let $f(\vec{x})$ where $\vec{x} = (x, y)$, denote the image intensity function of the given image in the domain Ω, and let $D \subset \Omega$ be the domain of inpainting. Let $u(\vec{x}, t)$ evolve in time to become a fully inpainted version of $f(\vec{x}) \in L^2(\Omega)$ under the following equation:

$$\frac{\partial u}{\partial t} = \Delta\left(-\epsilon \Delta u + \frac{1}{\epsilon} W'(u)\right) + \lambda_0 \mathbb{1}_{\Omega \setminus D}(f - u) \quad (1)$$

Where $\mathbb{1}_{\Omega\backslash D}(\vec{x})$ is the characteristic function of the complement of the inpainting domain

$$\mathbb{1}_{\Omega\backslash D}(\vec{x}) = \begin{cases} 0 & if\ \vec{x} \in D \\ 1 & if\ \vec{x} \in \Omega\backslash D \end{cases} \tag{2}$$

The constant $\lambda_0 \gg 1$ maintains the inpainted image close to the original image in $\Omega\backslash D$. The function $W(u)$ in Eq. (1) is a double well potential function with wells at $u = 0$ and $u = 1$, as binary images are only considered. In the current discussion, the double well potential function $W(u) = u^2(u-1)^2$ is used, though, the use of other functions is also possible.

TV-H⁻¹ Inpainting Model

The Total Variation of a $C^1(\overline{\Omega})$ function u defined on a bounded open set $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class C^1 can be expressed as

$$V(u,\Omega) = \int_{\Omega} |\nabla u(\vec{x})|\, d\vec{x} \tag{3}$$

The Total Variation based noise removal algorithm proposed by Rudin et al. [2] for a given image f is a L^2 gradient flow of the Total Variation functional, which, along with the regularizing term, generates the evolution equation

$$\frac{\partial u}{\partial t} = \nabla.\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda_0(f-u) \tag{4}$$

Since the Cahn-Hilliard inpainting model is only for binary images, Burger made a generalization of gray value images et al. [7], termed the TV-H^{-1} inpainting model based on similar lines with the Cahn-Hilliard equation. This model is termed TV-H^{-1} inpainting model and it follows the evolution equation

$$\frac{\partial u}{\partial t} = -\Delta\left(\nabla.\left(\frac{\nabla u}{|\nabla u|}\right)\right) + \lambda_0 \mathbb{1}_{\Omega\backslash D}(f-u) \tag{5}$$

In practice, to avoid dividing by 0 in Eq. (4) and (5), $\sqrt{|\nabla u|^2 + \delta^2}$ is used instead of $|\nabla u|$. Thus, the double well potential function has been dropped which makes the equation suitable for grayscale images. But, from the experimental results in [7], we can see that a better model is needed to make the edges even more smooth.

Tensor Based Anisotropic Diffusion

The heat equation is

$$\frac{\partial u}{\partial t} = \nabla.(c\nabla u) \tag{6}$$

Where c is a constant. Here, the flux $\vec{j} = -\,c\nabla u$ is always parallel to ∇u. If there is a need to orient the flux towards interesting features, then the flux should be $\vec{j} = -\,D\nabla u$, where $D \in \mathbb{R}^{2x2}$ is the Diffusion Tensor [9], [10]. In that case the diffusion equation becomes

$$\frac{\partial u}{\partial t} = \nabla.(D\nabla u) \tag{7}$$

where

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

Structure Tensor

A simple structure descriptor is given by ∇u_σ, the gradient of a Gaussian smoothed version of u:

$$K_\sigma(\vec{x}) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}} \tag{8}$$

$$u_\sigma(\vec{x},t) = K_\sigma(\vec{x}) * u(\vec{x},t) \tag{9}$$

The standard deviation σ denotes the noise scale, since it makes the edge detector ignorant of details smaller than $O(\sigma)$. The ∇u_σ is useful for detecting edges. To make the structure descriptor invariant under sign changes, we may replace ∇u_σ by its tensor product

$$J_0(\nabla u_\sigma) = \nabla u_\sigma \otimes \nabla u_\sigma = \nabla u_\sigma \nabla u_\sigma^T = \begin{bmatrix} u_{\sigma_x}^2 & u_{\sigma_x}u_{\sigma_y} \\ u_{\sigma_x}u_{\sigma_y} & u_{\sigma_y}^2 \end{bmatrix} \tag{10}$$

The matrix $J_0(\nabla u_\sigma)$ has eigenvalues $|\nabla u_\sigma|^2$ and 0 and corresponding eigenvectors $v_1 \,||\, \nabla u_\sigma$ and $v_2 \perp \nabla u_\sigma$.

EDGE ENHANCING DIFFUSION TENSOR

A diffusion tensor can be created which will enhance diffusion along the edges, i.e., along $v_2 \perp \nabla u_\sigma$ and minimize diffusion along $v_1 \,||\, \nabla u_\sigma$, since $|\nabla u_\sigma|^2$ will have a high value across the edges. Thus, the eigenvalues are chosen as

$$\mu_1 = e^{\frac{-|\nabla u_\sigma|^2}{k^2}} \qquad (11)$$
$$\mu_2 = 1$$

The diffusion tensor D is now constructed as

$$D\big(J_0(\nabla u_\sigma)\big) = [v_1 \quad v_2]\begin{bmatrix}\mu_1 & 0\\ 0 & \mu_2\end{bmatrix}\begin{bmatrix}v_1^T\\ v_2^T\end{bmatrix} \qquad (12)$$

where v_1 and v_2 are the orthogonal eigenvectors of $J_0(\nabla u_\sigma)$. These are discussed in detail in [9], [10].

EDGE ENHANCING ANISOTROPIC DIFFUSION

From Eq. (7), (10), (11) and (12), the Edge Enhancing Anisotropic Diffusion equation is formulated in [9], [10] as

$$\frac{\partial u}{\partial t} = \nabla.\left(D\left(J_\rho(\nabla u_\sigma)\right)\nabla u\right) \qquad (13)$$

$$u(\vec{x},0) = f(\vec{x})$$

III. PROPOSED INPAINTING MODEL – EDGE ENHANCING TV-H^{-1} EQUATION

The proposed model is TV-H^{-1} inpainting equation coupled with the edge enhancing diffusion tensor thus generating the equation

$$\frac{\partial u}{\partial t} = -\Delta\left(\nabla.\left(D\left(J_\rho(\nabla u_\sigma)\right)\frac{\nabla u}{|\nabla u|}\right)\right) + \lambda_0 \mathbb{1}_{\Omega\setminus D}(f-u) \qquad (14)$$

The boundary conditions in Eq. (14) are set to

$$\nabla u.\vec{n} = \nabla(\Delta u).\vec{n} = 0 \; on \; \partial\Omega \qquad (15)$$

IV. EXPERIMENTAL RESULTS AND DISCUSSION

The results of the experiments will be compared against other popular PDE-based image inpainting methods like the Transport Inpainting model proposed by Bertalmio et al. [1], TV-H^{-1} model proposed by Burger et al. and Cahn-Hilliard Perona Malik model proposed by Zou [8]. The results have been compared quantitatively based on four metrics: Mean Square Error (MSE), Peak Signal to Noise Ratio (PSNR), Structural Similarity Index Measure (SSIM) and Relative L^2 Error as discussed in [11]. The numerical schemes used were in line with what is discussed in [12]. The grayscale images are shown in Figure 1 and the results with two types of damages shown in Figure 2. The parameter settings are mentioned in Table 1, the quantitative error estimates in Table 2.

a. Cameraman   b. Lena   c. Peppers

FIGURE 1: Three grayscale testing images

Image	Method	Parameters settings
Cameraman #1	Transport	ε = 10^{-10}, M = 40, N = 2
	TV-H^{-1}	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 5
	Cahn-Hilliard Perona-Malik	ε = [200, 0.8], C_1 = 800, $C_2 = \lambda$ = 350
	Proposed model	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 10, k = 0.1, σ=1
Cameraman #2	Transport	ε = 10^{-10}, M = 40, N = 2
	TV-H^{-1}	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 5
	Cahn-Hilliard Perona-Malik	ε = [200, 0.8], C_1 = 500, $C_2 = \lambda$ = 280
	Proposed model	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 10, k = 0.1, σ=1
Lena #1	Transport	ε = 10^{-10}, M = 40, N = 2
	TV-H^{-1}	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 5
	Cahn-Hilliard Perona-Malik	ε = [200, 0.8], C_1 = 650, $C_2 = \lambda$ = 250
	Proposed model	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 10, k = 0.1, σ=1
Lena #2	Transport	ε = 10^{-10}, M = 40, N = 2
	TV-H^{-1}	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 5
	Cahn-Hilliard Perona-Malik	ε = [200, 0.8], C_1 = 500, $C_2 = \lambda$ = 250
	Proposed model	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 10, k = 0.1, σ=1
Peppers #1	Transport	ε = 10^{-10}, M = 40, N = 2
	TV-H^{-1}	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 5
	Cahn-Hilliard Perona-Malik	ε = [200, 0.8], C_1 = 800, $C_2 = \lambda$ = 300
	Proposed model	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 10, k = 0.1, σ=1
Peppers #2	Transport	ε = 10^{-10}, M = 40, N = 2
	TV-H^{-1}	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 5
	Cahn-Hilliard Perona-Malik	ε = [200, 0.8], C_1 = 500, $C_2 = \lambda$ = 70
	Proposed model	δ = 0.01, C_1 = 1000, $C_2 = \lambda$ = 10, k = 0.1, σ=1

TABLE 1: Parameters settings

Image	Method	Mean Square Error	PSNR	SSIM	Relative L^2 Error	CPU Time (secs)
Cameraman #1	Transport	0.011	19.5683	0.6966	0.0839	222.063
	TV-H^{-1}	0.0029	24.1291	0.9191	0.0462	109.873
	Cahn-Hilliard Perona Malik	0.0038	24.2502	0.8652	0.0503	337.998
	Proposed model	0.0021	26.7325	0.9249	0.0452	110.377
Cameraman #2	Transport	0.0104	19.8391	0.6778	0.0799	226.034
	TV-H^{-1}	0.0019	25.4859	0.9248	0.0279	117.229
	Cahn-Hilliard Perona Malik	0.0033	24.8516	0.8826	0.0421	342.74
	Proposed model	0.0016	27.9891	0.9447	0.0261	113.8
Lena #1	Transport	0.0128	18.9296	0.6364	0.1161	226.198
	TV-H^{-1}	0.0018	26.1292	0.9061	0.0395	101.974
	Cahn-Hilliard Perona Malik	0.0034	24.7148	0.8364	0.0538	348.554
	Proposed model	0.0018	27.7685	0.9214	0.0383	122.231
Lena #2	Transport	0.0115	19.4052	0.6302	0.1021	226.318
	TV-H^{-1}	0.0036	24.3243	0.9081	0.0486	102.881
	Cahn-Hilliard Perona Malik	0.0052	22.8038	0.8338	0.0538	340.31
	Proposed model	0.0034	24.7794	0.9152	0.0473	126.867
Peppers #1	Transport	0.0115	19.3851	0.6519	0.0846	226.669
	TV-H^{-1}	0.0018	27.4839	0.9031	0.0549	108.629
	Cahn-Hilliard Perona Malik	0.0025	26.0352	0.8737	0.0362	345.192
	Proposed model	0.0017	28.6112	0.9216	0.0359	107.088
Peppers #2	Transport	0.0153	18.1539	0.6002	0.0986	226.457
	TV-H^{-1}	0.0011	29.6866	0.9365	0.0215	102.945
	Cahn-Hilliard Perona Malik	0.0025	26.0724	0.8857	0.0314	344.587
	Proposed model	0.0021	30.3249	0.9612	0.0205	113.724

TABLE 2: Report for Image Inpainting

FIGURE 2: Inpainting Results with different algorithms

V. CONCLUSION

In this paper, we introduced a new image inpainting model based on TV-H^{-1} coupled with Edge Enhancing Structure Tensor. We executed the proposed model on commonly used grayscale images. The numerical experiments show that the proposed inpainting model works better than most of the accepted PDE models for image inpainting.

VI. REFERENCES

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