Gallai Type Results For Inverse Dominating Sets

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Abstract - Kulli and Sigarakanthi introduced the concept of inverse dominating sets. A set $D \subseteq V$ is said to be a dominating set if every vertex in *V-D* is adjacent to at least one vertex in *D*. Let *D* be a minimum dominating set of *G*. If *V-D* contains a minimal dominating set say D_1, then D_1 is called the inverse dominating set with respect to *D*. The inverse domination number $\gamma'(G)$ is the order of the smallest inverse dominating set of *G*. Domkey and others characterized the graphs for which $\gamma + \gamma' = p$. We disprove the result by exhibiting an infinite class of graphs which does not obey the conditions stated in the Theorem. We also prove the Kulli – Sigarakathi Conjecture that $\gamma'(G) \leq \beta_0(G)$.

Index Terms -Domination number, inverse domination number, disjoint domination number, Independence number

I. INTRODUCTION

The terminologies and notations used here are as in Harary [2]. A *graph G(V,E)* consists of a non empty set V whose elements are called vertices (or points) and a possibly empty set E of unordered pairs (*u, v*) of distinct vertices whose elements are called edges (or lines). Then $|V| = p$ and |E|= q are called *order* and *size* of G respectively. Degree of a vertex *v*, denoted as $d(v)$ is the number of edges incident on *v*. Similarly, the degree of an edge $x = uv$, denoted as $d_e(x)$ is the number of edges adjacent to the edge *x*. Equivalently, $d_e(x) = d(u) + d(v) - 2$. The maximum degree, the minimum degree, the maximum edge degree and the minimum edge degree of G are respectively denoted by $\Delta(G),\ \delta(G),\ \Delta_e(G),\ \ \delta_e(G).$ If $\Delta(G) = \delta(G) = k$, G is said to be *k-regular*. A vertex v is called an *isolated vertex* if $d(v) = 0$ and a *pendant vertex* if $d(v) = 1$. An edge incident on a pendant vertex is called a *pendant edge*. For any $v \in V$, the set $N(v) = \{u \in V \mid uv \in X\}$ is the *open neighbourhood* of the vertex v; while the set $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of *v*. Similarly, for any edge $x = uv$, $N(x) = \{y \in X \mid$ *y* is adjacent to *x*} and $N[x] = N(x) \cup \{x\}$. For any set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ is called the *open neighbourhood* of the set *S*. A *complete* graph K_p has every pair of its p vertices adjacent. The *complement* $\overline{G}$ of a graph G has $V(\overline{G}) = V(G)$ and $uv \in X(\overline{G})$ if, and only if, $uv \notin X(G)$. In particular, $\overline{K}_p$ has p vertices and no edges. Also, $\chi(K_p) = p$ and $\chi(\overline{K}_p) = 1$. A *bipartite graph* G is a graph whose vertex set can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. If every vertex of V_1 is joined with every vertex of V_2, then G is said to be *complete bipartite* graph and we write $G = K_{m,\,n}$ where $|V_1| = m$ and $|V_2| = n$. In particular, a complete bipartite graph $K_{1,\,n}$ is called a *star*. For any bipartite graph *G*, $\chi(G) = 2$. Let $G_1 = (V_1,\ X_1)$ and $G_2 = (V_2,\ X_2)$. Their *union* $G_1 \cup G_2$ has $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$. Their *join* $G_1 + G_2$ consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2. In particular, $K_{m,\,n} = \overline{K}_m + \overline{K}_n$. The *complete n-partite graph* $K_{p_1,\,p_2,\,\ldots,\,p_n}$ is defined as the iterated join of $\overline{K}_{p_1} + \overline{K}_{p_2} + \ldots\ldots + \overline{K}_{p_k}$. A *wheel* W_n invented by the eminent graph theorist W.T. Tutte, is defined as $K_1 + C_{n-1}$. For any graph with p vertices we have $G \cup \overline{G} = K_p$. The *Line graph* $L(G)$ of a graph G has vertex set as the edges of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges are adjacent in G.

A set $D \subseteq V$ is a *dominating set* of a graph *G* = (*V*, *E*), if every $v \in V\text{-}D$ is adjacent to some $u \in D$. The *domination number* $\gamma = \gamma(G)$ of *G* is the minimum cardinality of a dominating set of *G*. This concept is well studied in [3]. The concept of inverse domination is introduced by V.R.Kulli and S.C. Sigarakati [7]. Let *D* be a γ- set of *G*. If *V-D* also has a dominating set D_1 then D_1 is called the inverse dominating set of *G* with respect to *D*. The inverse domination number $\gamma'(G)$ is the order of a smallest inverse dominating set. If *D* is a minimal dominating set then *V-D* is also a dominating set of *G*. Therefore every graph has an inverse dominating set. It is observed that $\gamma(G) \leq \gamma'(G)$ and $\gamma(G) + \gamma'(G) \leq p$. For any graph *G* with *n*>2, a vertex *v* is said to be a pendent vertex if deg (*v*)=1 and any vertex adjacent to a pendent vertex is called a support or stem. A set $D \subseteq V$ is said to be independent if no two vertices in *D* are adjacent. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set of *G*. We say that an edge *x* and a vertex *v* cover each other if *x* is incident on *v*. A set $D \subseteq V$ is said to be a vertex cover if every edge in *G* is covered by some vertex in *D*. The *vertex covering number* $\alpha_0(G)$ is the minimum cardinality of a vertex cover of *G*. A set $S \subseteq V$ is a vertex cover if, and only if, *V-S* is an independent set. A set $L \subseteq X$ is said to be an *edge cover* if the edges of L cover all the vertices of G. The *edge covering number* $\alpha_1 = \alpha_1(G)$ is the minimum cardinality of an edge cover of G. A set $L \subseteq X$ is said to be *edge independent* if no two edges are adjacent. The *edge independence number* $\beta_1 = \beta_1(G)$ is the maximum cardinality of an edge independent set of G. An edge independent set is also called a *matching*.

For the graph given in the Fig.1.1, $\alpha_0(G) = 2 = \beta_1(G)$ and $\beta_0(G) = \alpha_1(G) = 4$. An α_o-set is $S_1 = \{v_2, v_5\}$ and β_o-set is $S_2 = \{v_1, v_3, v_4, v_6\}$. β_1- set is $\{(v_1, v_2), (v_5, v_6)\}$. An α_1-set is. $\{(v_1, v_2), (v_4, v_5), (v_3, v_5), (v_5, v_6)\}$

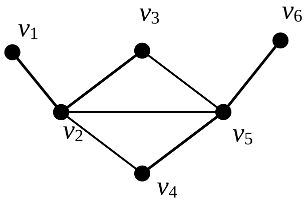

Fig. 1.1

The independence number and vertex covering number are related by Gallai's Theorem. Similar result holds for edge independent sets and edge covering.

Theorem 1.1 (Gallai). *For any isolate free graph G,*

$\alpha_o(G) + \beta_o(G) = p$

$\alpha_1 + \beta_1 = p$

The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. A *subdivision* of an edge *uv* is obtained by replacing the edge *uv* with a new vertex *w* and the edges *uw* and *vw*. A *spider* is a tree on $2n + 1$ vertices obtained by subdividing each edge of a star $K_{1,n}$. A *wounded spider* is the graph formed by subdividing at most $n-1$ edges of a star $K_{1,n}$. Thus, K_1, $K_{1,n}$ and the corona $(K_{1,n}) \circ (K_1)$ are the examples of wounded spider. A *caterpillar* *C* is a tree, the deletion of whose end vertices results in a path called *spine* of *C*. Fig.1.2 provides the examples of a corona $C_3 \circ C_3$, a spider, a wounded spider.

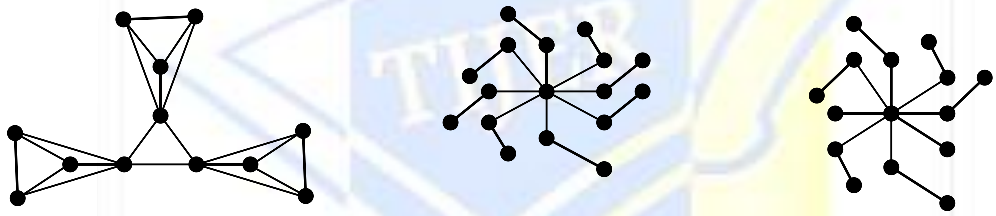

Fig. 1.2 Corona $C_3 . C_3$, Spider and

Disjoint domination number (defined by Hedetnimi et al. [4]) $\gamma\gamma(G)$ is defined as min{ $|S_1| + |S_2|$; S_1, S_2 are disjoint dominating sets of *G*}and wrongly observed that $\gamma\gamma(G) \le \gamma + \gamma'$. Since the sum $|S_1| + |S_2|$ is minimum if and only if both S_1 and S_2 are minimum and S_1 and S_2 are disjoint implies that S_1 is a γ- set and S_2 is a γ'-set. Hence $\gamma\gamma(G) = \gamma + \gamma'$. They call *G* is $\gamma\gamma$-minimum if $\gamma\gamma(G) = 2\gamma(G)$ and *G* is $\gamma\gamma$-maximum if $\gamma\gamma(G) = p$. It is observed that inverse domination umber and disjoint domination numbers coincide. For example , $\gamma^{!}(C_3 . C_3) = 3 = \gamma\gamma(C_3 . C_3)$and for the spider S shown in the Fig 1.2 , $\gamma^{!}(S) = 9 = \gamma\gamma(S)$.

In the following Theorem $\gamma\gamma$-maximum graphs are characterized.
Theorem 1.2 [4]**.** *A connected graph G is $\gamma\gamma$-maximum if and only if either G is C_4 or every vertex is a leaf or a stem.*

We disprove Theorem 1.2 by exhibiting a counter example of an infinite class of trees. A *spider* is a tree obtained by subdividing each edge of the star $K_{1,n}$. Any pendent edge of a spider is called *leg* of the spider. If every support in a spider has at least two legs then it is called *multi lgged spider*. The maximum degree vertex is called the *heart* of the spider. For any multi legged spider, the set *S* of all the supports forms a γ-set and *V-S* is the next minimum dominating set disjoint from *S*. Hence $\gamma\gamma(G) = p$. Thus any multi legged spider is a $\gamma\gamma$-maximum graph. But the heart of a multi legged spider is neither a leaf nor a stem.

II GALLAI TYPE RESULTS FOR INVERSE DOMINATION

In fact in the above theorem Hedetnimi et.al tries to show when $\gamma\gamma(G) = \gamma + \gamma' = p$. This is a result similar to Gallai's Theorem. This question is answered in the next result. Domke et al. [1] characterized the graphs for which $\gamma(G) + \gamma'(G) = p$ in the following Theorem.

Theorem 2.1[1]. *Let G be a graph with* $\delta(G) \geq 2$. *Then* $\gamma(G)+\gamma'(G) = p$ *if and only if* $G=C_4$.

Theorem 2.2[1]. *Let G be a connected graph with* $n \geq 3$ *and* $\delta(G)=1$. *Let L be the set of all leaves and* $S = N(L)$ *(stems). Then* $\gamma(G)+\gamma'(G) = p$ *if and only if the following conditions hold:*

(i) *V-S is an independent set.*

(ii) *For every vertex* $x \in V-(S \cup L)$ *every stem in* $N(x)$ *is adjacent to at least two leaves.*

We now disprove Theorem 2.2 by exhibiting an infinite class of graphs which do not satisfy the condition (ii) of the Theorem.

For the Corona $G = K_3 \circ K_1$ shown in the Fig 2.1 $\gamma(G)=3$ and $\gamma'(G)=3$ and $\gamma(G)+\gamma'(G) = 6= p$ but this graph does not satisfy the condition (ii) of the Theorem 4. In fact the infinite class of Corona $H \circ K_1$, for any connected graph *H*, does not satisfy the condition (ii) of the Theorem.

Observe that the Domke's result fails when $\gamma(G)=\frac{p}{2}$. Hence we restate the Domke's result as follows.

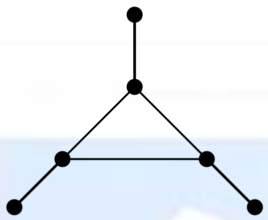

Fig.2.1. The Corona $K_3 \cdot K_1$

Theorem 2.2. *Let G be a connected graph with* $\delta(G)=1$ *and* $\gamma(G)< \frac{p}{2}$. *Let L be the set of all leaves and* $S = N(L)$ *(stems). Then* $\gamma(G)+\gamma'(G) = p$ *if and only if the following conditions hold:*

(i) *V-S is an independent set.*

(ii) *Every stem is adjacent to at least two leaves.*

If $\gamma(G)=\frac{p}{2}$ then there exists a γ-set *S* such that $\frac{p}{2} = |S| = \gamma \leq \gamma'$. Also $\gamma' \leq p-\gamma = \frac{p}{2}$. Hence $\gamma'(G)=\frac{p}{2}$. Thus we have the following Theorem complementing Domke's result.

Theorem 2.3. *Let G be a connected graph with* $\delta(G)=1$. *Let L be the set of all leaves and* $S = N(L)$ *(stems). Then* $\gamma(G)+\gamma'(G) = p$ *if and only if either* $\gamma(G)=\frac{p}{2}$ *or the following two conditions hold:*

(i) *V-S is an independent set.*

(ii) *Every stem is adjacent to at least two leaves.*

The graphs for which $\gamma(G)=\frac{p}{2}$ is characterized independently by Payan and Xuong [8] and by Fink et.al [5].

Theorem 2.4 [5]. *For any graph G, without isolates ,* $\gamma(G)=\frac{p}{2}$ *if and only if G* is the Corona $H \circ K_1$ or the cycle C_4.

The next Corollary completely characterize the class of graphs for which $\gamma(G)+\gamma'(G) = p$

Corollary 2.4.1. *Let G be a graph without isolates. Then* $\gamma(G)+\gamma'(G) = p$ *if and only if, each component of G is* C_4 *or the Corona* $H \circ K_1$ *for some connected graph H or the graph satisfying the conditions (i) and (ii) stated in the Theorem* 6.

We next give another necessary and sufficient condition for $\gamma(G)+\gamma'(G) = p$.

Theorem 2.5. *Let G be a graph of order p without isolates. Then* $\gamma(G)+\gamma'(G) = p$ *if and only if* $\gamma(G)= \alpha_0(G)$ *and* $\gamma'(G)= \beta_0(G)$.

Proof. Suppose $\gamma(G)+\gamma'(G) = p.$

Case (i). $\delta(G) \geq 2$: Then by Theorm 3, *G* is C_4 and for this graph one can easily verify that $\gamma(G) = \alpha_0(G)$ *and* $\gamma'(G) = \beta_0(G)$

Case (ii). $\delta(G)=1$: Then from the Theorem 8, G is either the Corona $H \circ K_1$ or the graph satisfying conditions (i) and (ii) said in the Theorem 6.

Subcase (i). Let $G = H \circ K_1$ and $D=V(H)$. Then $\gamma(G) = \alpha_o(G)=|D|$ and $\gamma'(G)= \beta_o(G)= |V-D|$. Hence the result.

Subcase (ii). Let G be the graph satisfying conditions (i) and (ii) said in the Theorem 5.

Let S be the set of all stems and L be the set of all leaves. Since $N(S)=V$ we have S is a dominating set of G and $\gamma(G) \leq |S|$. Further in order to dominate the vertices in L at least $|N(L)|=|S|$ vertices are required, hence $\gamma(G) \geq |S|$. Thus S is a γ-set of G and V-S is a γ'- set of G. From condition (i) of Theorem 6, V-S is an independent set and hence S is a covering of G. Therefore $\alpha_o(G) \leq |S|=\gamma(G)$.

But it is well known that $\gamma(G) \leq \alpha_o(G)$. Thus $\gamma(G)= \alpha_o(G)$. Since S is a minimum covering V-S is a maximum independent set. Therefor $\beta_0 = |V-S| = p-\alpha_0 = p-\gamma = \gamma'$.

Remark 1. Note that both the conditions in the Theorem 9, $\gamma(G)= \alpha_o(G)$ and $\gamma'(G)= \beta_o(G)$ are essential. For example for the graph$G=K_{2,3}$, $\gamma(K_{2,3}) = \alpha_o(K_{2,3})=2$. But $\gamma'(K_{2,3})=2\neq3= \beta_o(K_{2,3})$. Hence $\gamma(K_{2,3})+\gamma'(K_{2,3})= 4 \neq5= p$. Again for the complete graph K_p, $\gamma'(K_p)=1= \beta_o(K_p)$. But $\gamma(K_p)= 1\neq p$-$1= \alpha_o(K_p)$. Hence $\gamma(K_p)+\gamma'(K_p) =2 \neq p$.

We now prove the Kulli - Sigarakanthi conjecture. Sigarakanthi and Kulli [7] proved that for any graph G, $\gamma'(G)\leq\beta_o(G)$. But Hedetnimi et.al [4] found the proof is with some error and could not give a correct proof or even not disproved the result. They called it as Kulli - Sigarakanthi conjecture in [4]. We give a simple but an elegant proof.

Proposition 2.6. For any graph without isolates, $\gamma'(G)\leq \beta_o(G)$.

Proof. Let D be any maximum independent set of G. Then V-D is a minimum covering of G. Every minimum covering contains a minimum dominating set. Let $S\subseteq V$-D be a minimum dominating set of G. Since D is a maximum independent set we have D is also a dominating set of G. Therefore D is an inverse dominating set of G with respect to S. Hence $\gamma'(G)\leq |D|=\beta_o(G)$.

II. CONCLUSIONS

We proved the Kulli- Sigarakanthi conjecture that inverse domination number is atmost independence number of a graph. Also we characterized the graphs satisfying Gallai's Theorem type results.

III. REFERENCES

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