Review of Fixed Point Theory

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Abstract - Fixed point theory is developed to find out the fixed point for selfmaps in Metric Space. The famous Mathematician H. Poincare in 1912, Banach in 1922, Browder in 1965, and Kannan in 1969 obtained more general fixed point results by developing fixed point theorems. And it was further generalized by Dolhare U. P. and Nalawade by using some contractive conditions to find out fixed point. We have also established fixed point theorems in complete Metric Space, which is a new generalized result for fixed point theory.

Index Terms - Fixed Point, Banach Fixed Point Theorem, Contraction Mapping, Non-expansive mapping.

I. INTRODUCTION

The Fixed Point technique is one of the most important and powerful tools used for solving the nonlinear operator equations which is a major and core part of nonlinear functional analysis. In the Fixed Point Theory the main problem is to find out the points which are invariant under the action of the mapping under consideration. The topic about finding the fixed points of the mapping is the most active area of research work in analysis and topology since long time.

The importance of study of nonlinear problems was first recognized by the famous French Mathematician H. Poincare (1854-1912). He predicted that the future Mathematics will mainly deal with the nonlinearity. Since then several methods are developed and employed in the study of nonlinear equations.

The concept of Metric Space is formulated by the Mathematician M. Frechet in 1906. He firstly introduced the notion of Metric Space in 1906 and the Mathematician Hausdroff further developed it in 1914.

The Fixed Point Theory of fixed point theorems is developed by Picard iteration method. Mathematician Banach [1], in 1922, first proved the basic fixed point theorem called the Banach Contraction Mapping Principle. Kannan [5] further developed the Fixed Point Theory and since then several fixed point theorems have been proved in Metric Spaces.

Metric Spaces are the sets in which a notion of 'distance between pair of points' is defined and they provide the general setting in which the concepts like Convergence and Continuity are studied.

Kannan [5] and Rhoades [7] generalized the Metric Space and some fixed point theorems. These spaces are developed for the study of fixed point theorems under different contraction conditions, which are more useful for the formation of fixed point theorems.

II. FIXED POINTS AND FIXED POINT THEORY

The Fixed Point Theory is one of the most important and major topics of the nonlinear functional analysis. It has applications also in other branches of Mathematics mainly for proving the existence and uniqueness theorems to nonlinear problems. Fixed Point theorems are formulated by using many topological spaces such as Metric, D-Metric, Hausdorff, Banach, Hilbert, and locally convex space.

Definition 2.1: A fixed point of a function is a point which is mapped to itself by the function, that is, x is a fixed point of the function f if and only if $f(x) = x$. For example, if f is defined on the set of real numbers by $f(x) = x^2 + 2x - 12$ then 3 is a fixed point of the function f, as $f(3) = 3$.

Example 2.1: Consider the integral equation

$$X(s) = x_1(s) + \int_a^b f(s,u,x(u))\,du, \qquad s \in [a,b],$$

where x_1 is a given continuous real-valued function on $[a,b]$.

Let us denote the set of all real-valued continuous functions on $[a,b]$ by S and

$$F(x)_S = x_0 + \int_a^b f(s,u,x(u))\,du, \qquad x \in S, s \in [a,b].$$

If $f(s,u,x(u))$ is an integrable function of u, for each $s \in [a,b]$, on $[a,b]$ and the function $\int_a^b f(s,u,x(u))\,du$ is a continuous function of s on $[a,b]$, then $f(x) \in S$. Then obviously x is a solution of this integral equation if and only if it is a fixed point of the function F.

For studying fixed points, a well-known classical result about the existence and uniqueness of a fixed point of a contraction is very useful which is as below:

A function f from a metric space X to itself is a contraction if it satisfies

$$d(f(x), f(y)) \leq \lambda\, d(x, y);$$
$$\forall\, x, y \in X \text{ and for some } \lambda \text{ with } 0 \leq \lambda < 1.$$

We have applied this result to the solution of an initial value problem after converting it to an integral equation.

In fixed point theory, the main problem is to find the points which remain invariant under the action of the mapping considered, that is to obtain the solution of functional equation $f(x) = x$ in the appropriate function spaces.

Example 2.2: Consider the function $f: R \to R$ defined by $f(x) = x^2,\ x \in R$. This function has only two fixed points and they are 0 and 1 as $f(0) = 0$ and $f(1) = 1$.

III. SOME FIXED POINT THEOREMS

In this section the Brouwer's and Schauder's Fixed Point Theorems have been proved. These two theorems are the fundamental results in the area of Fixed Point Theory. Brouwer proved his result in 1910, but its slightly different version was proved in 1886 by Poincare and it was subsequently rediscovered in 1904 by Bho. There are many proofs of Brouwer's theorem - topological, analytical and degree theoretic. Here the proofs due to Kannan [5] have been represented.

Definition 3.1: A topological space X is said to possess the Fixed Point Property if every continuous function from X into itself has a fixed point. A finite closed interval $[a, b]$ and the closed unit disk in the plane has the fixed point property.

Theorem 3.1: Let X and Y be topological spaces. If X has the fixed point property and Y is homeomorphic to X then Y has the fixed point property.

Corallary 3.1: If $B = B(0, R),\ R > 0$ and $f: B \to B$ is a continuous map, then f has a fixed point.

Brouwer has also proved the fixed point theorem for compact and convex set for continuous map as below:

Theorem 3.2: (Brouwer [2]) Let $K \subset R^n$ be a compact and convex set and $f: K \to K$ be a continuous map, then f has a fixed point.

Theorem 3.3: (Browder and Browder-Petryshyn [3, 4]) Let H be a Hilbert space and K be a closed convex and bounded subset of H. If $F: K \to K$ is a non - expansive mapping then F has a fixed point.

Theorem 3.4: Let K be a non-empty bounded closed and convex subset of a reflexive Banach space X and suppose that K has a normal structure. If F is a non-expansive mapping of K into itself, then F has a fixed point.

Theorem 3.5: Let H be a Hilbert space and let K be a closed star-shaped subset of H. And suppose that $F: K \to H$ is non-expansive satisfying $F(\partial K) \subset K$ and there exists $x_0 \in K$ with iterative bounded sequence $\{F^n(x_0)\}$. Then F has a fixed point.

Theorem 3.6: (Brouwer [2]) Let $f: B \to B$ be a continuous map. Then f has a fixed point.

Proof: If f has no fixed point then $f(x) - x \neq 0$ for every x in B. But as B is compact, therefore there exists a constant $c > 0$ such that $|f(x) - x| \geq c$ for every x in B. Then we can find a function $\phi: B \to B,\ \phi$ of class c^2, such that $|\phi(x) - f(x)| < \frac{c}{2}$ for every $x \in B$. So that we get $|\phi(x) - x| \geq \frac{c}{2}$ for every $x \in B$. Again we define $\psi : B \to S$ as, $\psi(x)$ is the intersection of the line from $\phi(x)$ to x with S. Now as $\phi(x) \neq x$, therefore this line and hence the function $\psi(x)$ is always well defined. The map $x \to \psi(x)$ is of class c^2, certainly $\psi(x) = \lambda x + (1 - \lambda)\phi(x),\ \lambda \geq 1,\ |\psi(x)| = 1$.

Now

$$|\psi(x)|^2 = \lambda^2 |x|^2 + (1 - \lambda)^2 |\phi(x)|^2 + 2\,\lambda\,(1 - \lambda)\big(x, \phi(x)\big) = 1$$
$$\lambda^2 |x - \phi(x)|^2 + 2\,\lambda\,\big(x - \phi(x), \phi(x)\big) + |\phi(x)|^2 - 1 = 0.$$

This has exactly one root $\lambda \geq 1$ again as $x - \phi(x) \neq 0$, therefore the map $x \to \lambda(x)$ is c^2 and therefore the map $x \to \psi(x)$ is also c^2. But if x is in S, then obviously $\psi(x) = x$ and thus $\psi: B \to S$ is a c^2 retraction which is a contradiction. Hence the function f has a fixed point.

Theorem 3.7: If Y has the fixed point property and X is retract of Y then X has the fixed point property.

Proof: Let R be a retraction mapping of Y onto X and F be a continuous mapping of X into itself, then FR is a continuous mapping of Y into X. Now as FR maps Y into itself, therefore there is a fixed point u, that is $FRu = u$. But as $u \in X$, therefore $Ru = u$, which implies $Fu = u$.

Theorem 3.8: (Schauder [8]) Let V be a Banach space and let $K \subset V$ be a convex, compact set. If $f: K \to K$ is a continuous map, then f has a fixed point.

Proof: For $\epsilon > 0$, let V_ϵ be a finite dimensional subspace of a Banach space V and consider a continuous map $h_\epsilon : K \to V_\epsilon$ such that $\|h_\epsilon(v) - v\| \leq \epsilon$ for every $v \in K$. Now as K is convex, therefore $h_\epsilon(K) \subset K$. Let K_ϵ be the convex hull of $x_1, x_2, \ldots, x_n$. Then clearly $K_\epsilon \subset K$. Again define $f_\epsilon : K_\epsilon \to K_\epsilon$ by $f_\epsilon(x) = h_\epsilon(f(x))$. Now as K_ϵ is a compact convex set in the finite dimensional space V_ϵ, from the Brouwer's theorem it follows that it has a fixed point $x_\epsilon \in K_\epsilon \subset K$.

Thus $x_\epsilon = h_\epsilon(f(x_\epsilon))$. As K is compact, a convergent subsequence can be extracted. Let $x_\epsilon \to x \in K$ as $\epsilon \to 0$.

Now we have

$$\|x - f(x)\| \le \|x - x_\epsilon\| + \|x_\epsilon - f(x_\epsilon)\| + \|f(x_\epsilon) - f(x)\|$$

the first term on the right tends to zero as $\epsilon \to 0$. Again as f is continuous, the last term on the right tends to zero as $\epsilon \to 0$. Also

$$\|x_\epsilon - f(x_\epsilon)\| = \|h(f(x_\epsilon)) - f(x_\epsilon)\| \le \epsilon$$

since $f(x_\epsilon) \in K$ by corollary 3.1. Thus $\|x - f(x)\|$ can be made arbitrarily small and therefore we get $x = f(x)$. Hence the mapping f has a fixed point.

Theorem 3.9: (Kirk and Ray [6]) Let K be a closed convex subset of uniformly convex Banach space X. Let $F: K \to K$ be a Lipschitzian Pseudo-contractive mapping. And suppose for some $a \epsilon K$ the set $G(a, Fa) = \{z \in K: \|z - a\| \ge \|z - Fa\|\}$ is bounded. Then F has a fixed point in K.

IV. MAIN RESULTS

We generalized above theorem 3.9 as follows:

Proof: Let us suppose that K is the Lipschitz constant. We choose $\alpha \in (0, 1)$ such that $\alpha k < 1$. For each $y \in K$ let $F_y : K \to K$ is defined by $F_y(x) = (1 - \alpha)\, y + \alpha\, T_x$. Now as F_y is a contraction mapping, therefore it has a fixed point, say $T_\alpha(y)$ for each $y \in K$, that is $T_\alpha(y) = (1 - \alpha)\, y + \alpha\, F\big(T_\alpha(y)\big)$, $y \in K$.

If $r > 0$, then $\|u - v\| \le \|(1 + r)(u - v) - r(Fu - Fv)\|$. Further if r is chosen small such that $\alpha(1 + r) > r$, then we get $\|T_\alpha(u) - T_\alpha(v)\| \le \|u - v\|$. So $T_\alpha: K \to K$ is non-expansive on K. Again the existence of a fixed point of T_α gives the fixed point of F. This is accomplished by showing $G(a, T_\alpha\,(a))$ is bounded for sufficiently small $\alpha \in (0,1)$. Now if $G(a, T_\alpha\,(a))$ is bounded for some $a \in K$ then T_α has a fixed point in K.

Theorem 4.1: Let K be a bounded, convex and closed subset of a uniformly convex Banach space X and let $F: K \to X$ be a non-expansive weakly inward mapping. Then F has a unique fixed point.

Proof: Let us define F_n by

$$F_n(x) = (1 - \alpha)x_0 + \alpha_n\, F_x \text{ where } x_0 \in K,\ 0 < \alpha_n < 1,\ \text{and} \lim_{n\to\infty} \alpha_n = 1.$$

Then obviously each F_n is a contraction with Lipschitz constant α_n, (< 1) and therefore F_n has a unique fixed point, say x_n. Again as K is bounded, therefore

$$\|x_n - Fx_n\| = \left\|x_n - \frac{x_n}{\alpha_n} - \left(\frac{1}{\alpha_n} - 1\right) x_0\right\|$$

$$= \left(\frac{1}{\alpha_n} - 1\right) \|x_n - x_0\| \to 0 \text{ as } n \to \infty.$$

Now let x be the asymptotic centre of the sequence $\{x_n\}$ with respect to k, then we have

$$r(Fx) = \lim\sup_n \|Fx - x_n\|$$
$$\le \lim\sup_n \|Fx - Fx_n\|$$
$$\le \lim\sup_n \|x - x_n\| = r(x)$$

as F is non-expansive. Again as $Fx \in \overline{I_k(x)}$ and α is asymptotic centre of $\{x_n\}$ with respect to $\overline{I_k(x)}$, therefore by the uniqueness of the asymptotic centre we obtain $Fx = x$.

V. CONCLUSION

In this paper we have discussed and analysed mappings and Fixed Point Theory. We have observed that there are some points which remain invariant under the action of the mapping. Such points are called Fixed Points. They play a very important role in solving operator equations. Brouwer and Schauder shown that the continuous mappings have fixed points. Also the non-expansive mappings are useful in solving operator equations and in obtaining fixed points of mappings.

ACKNOWLEDGEMENT

We are thankful to the editors and referees for their valuable suggestions.

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