Existence Of Some Application's Of Fixed Point Theorem For A Nonlinear Differential Equation

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Abstract:- In this paper, We study a new family of mappings on [0, +∞) by relaxing the non-decreasing condition on the mappings and by using the properties of this new family we present some fixed point theorems for ⍰-⍰-contractive-type mappings in the setting of complete metric spaces. By applying our obtained results, we provide an existence theorem for a nonlinear differential equation.
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Introduction and Preliminaries

The fixed point theory has been continuously studied by many researchers since 1922 with the collaboration of Banach fixed point theorem. Fixed point theorems are very important tools for proving the existence and uniqueness of the solutions to various mathematical models. It can be applied to, for example, variational inequalities, optimization, and approximation theory.. It is well known that the contractive-type conditions are very indispensable in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach-Caccioppoli theorem which was published in 1922 . Later in 1968, Kannan studied a new type of contractive mappings. Since then, there have been many results related to mappings satisfying various types of contractive inequalities and references contained therein.

Samet et al. gives a new category of contractive-type mappings known as ⍰-⍰ contractive - type mappings. The results obtained by Samet et al. extended and generalized the existing fixed point results in the literature, in particular the Banach contraction principle. Salimi et al. and Karapinar and Samet generalized the ⍰-⍰ contractive-type mappings and obtained various fixed point theorems for this contractive mappings .

Most of papers have considered the ⍰-⍰ contractive-type mapping for a nondecreasing mapping ⍰ : [0, +∞) → [0, +∞) with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all ⍰ ∈ (0, +∞). The convergence of $\sum_{n=1}^{\infty} \psi^n(t)$ and nondecreasing condition for ⍰ are restrictive and it is a fact that such a mapping is differentiable almost everywhere and hence continuous why was one of our aims to write this paper in order to consider a family of mappings ⍰ : [0, +∞) → [0, +∞) by relaxing nondecreasing condition and the convergence of the series $\sum_{n=1}^{\infty} \psi^n(t)$. This paper is inspired and motivated by research works, we will introduce a new family of mappings on [0, +∞) and prove the fixed-point theorems for mappings using properties of this new family in complete metric spaces. By applying our obtained results, we also assure the fixed point theorems in partially ordered complete metric spaces and give the applications to ordinary differential equations.

In the rest of the paper, we introduce some notations and definitions that will be used .

Definition (1): - Let ⍰ : ⍰ → ⍰ and let ⍰ : ⍰ × ⍰ → [0, +∞). We say that ⍰ is ⍰-admissible if, for all ⍰, ⍰ ∈ ⍰, ⍰(⍰, ⍰) ≥ 1 implies ⍰(⍰⍰, ⍰⍰) ≥ 1.

In 2012, Samet et al. introduced the concept of ⍰-⍰contractive-type mappings, where ⍰ ∈ Ψ_1 and Ψ_1 = { ⍰ : ⍰ : [0, +∞)→ [0, +∞) is nondecreasing with lim n=1 to ∞ Σ ⍰⍰ (⍰) < ∞, ∀ ⍰ ∈ (0, +∞) } .

Lemma (2):- - Suppose that ⍰ : [0, +∞) → [0, +∞). If ⍰ is nondecreasing, then for each ⍰ ∈ (0, +∞), lim⍰ → ∞⍰⍰(⍰) = 0 implies that ⍰(⍰) < ⍰.

Definition(3):- Let (⍰, ⍰) be a metric space and let ⍰ : ⍰ → ⍰ be a mapping. We say that ⍰ is an ⍰-⍰-contractive mapping if there exist two functions ⍰ : ⍰ × ⍰ → [0, +∞) and ⍰ : [0, +∞) → [0, +∞) where ⍰ ∈ Ψ_1 such that

⍰ (⍰, ⍰) ⍰ (⍰⍰, ⍰⍰) ≤ ⍰ (⍰ (⍰, ⍰)) , (2) for all ⍰, ⍰ ∈ ⍰.

The authors assured the existence of the fixed point theorems for the mentioned mappings satisfying ⍰admissibility in the complete metric spaces.

Recently, Salimi et al. modified the concept of ⍰-admissibility.

Definition (4): -. Let ⍰ : ⍰ → ⍰ and ⍰ , ⍰ : ⍰ × ⍰ → [0, +∞). We say that ⍰ is ⍰-admissible with respect to ⍰ if, for all ⍰, ⍰ ∈ ⍰, ⍰(⍰, ⍰) ≥ ⍰(⍰, ⍰) implies ⍰(⍰⍰, ⍰⍰) ≥ ⍰(⍰⍰, ⍰⍰).

Remark (5):- If we suppose that ⍰(⍰, ⍰) = 1, for all ⍰, ⍰ ∈ ⍰.

Definition (6):- Let $T : X \to X$ and let $\alpha : X \times X \to [0, \infty]$. We say that T is α – admissible if , for all x, y ϵ X, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Salimi et al. proved the existence of fixed point theorems for generalized ⍰-⍰-contractive-type mappings where ⍰ ∈ Ψ_1. They also assure the fixed point theorems generalized ⍰-⍰-contractive-type mappings where ⍰ is a nondecreasing continuous mapping and ⍰(0) = 0.

In this work, we will introduce a new family of mappings on [0, +∞) without assuming the nondecreasing condition for ⍰ and prove the fixed point theorems for ⍰-⍰-contractivetype mappings using properties of this new family in complete metric spaces. We will use our result to obtain fixed point results in partially ordered complete metric spaces and to give an application to nonlinear differential equations.

Main Results

We now introduce a new family Ψ_2 of mappings and prove the existence of fixed point results for ⍰-⍰-contractive-type mappings where ⍰ ∈ Ψ_2.

Denote by Ψ_2 the family of mappings ⍰ : [0, +∞) → [0, +∞) such that

(i) ⍰ is an upper semicontinuous mapping from the right;

(ii) ⍰(⍰) < ⍰ for all ⍰ ∈ (0, +∞); (iii) ⍰(0) = 0.

Remark (7):- Since every nondecreasing mapping is differentiable almost everywhere, we observe that nondecreasing condition is closed to continuity and it is restrictive.

Example (8):-- The floor function ⍰(⍰) = ⌊⍰⌋ is upper semicontinuous function from the right and nondecreasing but is not continuous.

Theorem(9):- Let (⍰, ⍰) be a complete metric space and ⍰ ∈ Ψ_2. Suppose that ⍰ : ⍰ → ⍰ is a mapping satisfying the following conditions:

(i) ⍰ is ⍰-admissible with respect to ⍰;

(ii) if ⍰, ⍰ ∈ ⍰ and ⍰(⍰, ⍰) ≥ ⍰(⍰, ⍰), then ⍰(⍰⍰, ⍰⍰) ≤ ⍰(⍰(⍰, ⍰));

(iii) there exists $⍰_0$ ∈ ⍰ such that ⍰($⍰_0$, ⍰$⍰_0$) ≥ ⍰($⍰_0$, ⍰$⍰_0$);

(iv) ⍰ is continuous or if {$⍰_⍰$} is a sequence in ⍰ such that ⍰($⍰_⍰$, $⍰_{⍰+1}$) ≥ ⍰($⍰_⍰$, $⍰_{⍰+1}$) for all ⍰ ∈ N and $⍰_⍰$ → ⍰ ∈ ⍰ as ⍰ → ∞, and then ⍰($⍰_⍰$, ⍰) ≥ ⍰($⍰_⍰$, ⍰) for all ⍰ ∈ N.

Then, ⍰ has a fixed point.

Proof:- Since $⍰_0$ ∈ ⍰, there exists $⍰_1$ such that $⍰_1$ = ⍰$⍰_0$.Therefore, we can construct the sequence {$⍰_⍰$} in ⍰ such that $⍰_{⍰+1}$ = ⍰$⍰_⍰$, ∀⍰ ∈ N.

If $⍰_{⍰+1}$ = $⍰_⍰$, for some ⍰ ∈ N, then ⍰ has a fixed point. Assume that $⍰_⍰$ ≠ $⍰_{⍰+1}$ for all ⍰ ∈ N. Since ⍰($⍰_0$, $⍰_1$) = ⍰($⍰_0$, ⍰$⍰_0$) ≥ ⍰($⍰_0$, ⍰$⍰_0$) and ⍰ is ⍰-admissible with respect to ⍰, we obtain that

⍰ ($⍰_1$, $⍰_2$) = ⍰ (⍰$⍰_0$, ⍰$⍰_1$) ≥ ⍰ (⍰$⍰_0$, ⍰$⍰_1$) = ⍰ ($⍰_1$ x_2)

By continuing the process as above, we have

⍰ ($⍰_⍰$, $⍰_{⍰+1}$) ≥ ⍰ ($⍰_⍰$, $⍰_{⍰+1}$) , ∀⍰ ∈ N.

Applying (ii), we obtain that

⍰ ($⍰_⍰$, $⍰_{⍰+1}$) = ⍰ (⍰$⍰_{⍰-1}$, ⍰$⍰_⍰$) ≤ ⍰ (⍰ ($⍰_{⍰-1}$, $⍰_⍰$)) , for all ⍰ ∈ N. Since ⍰(⍰) < ⍰ for all ⍰ ∈ (0, +∞), we have

⍰ ($⍰_⍰$, $⍰_{⍰+1}$) ≤ ⍰ (⍰ ($⍰_{⍰-1}$, $⍰_⍰$)) < ⍰ ($⍰_{⍰-1}$, $⍰_⍰$) , for all $n \in \mathbb{N}$. Therefore, $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence. It follows that there exists $c \geq 0$ such that $\lim_{\eta\to\infty} d(x_n, x_{n+1}) = c$

We will prove that $c = 0$. Suppose that $c > 0$. Since ψ is upper semicontinuous from the right using , we have $c = \lim_{\eta\to\infty} \sup d(x_n, x_{n+1})$

$$\leq \lim_{\eta\to\infty} \sup \psi(d(x_{n-1}, x_n)) \leq \psi(c) < c,$$

which is a contradiction. Therefore,

$$\lim_{\eta\to\infty} d(x_n, x_{n+1}) = 0.$$

This implies that for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$d(x_{nk}, x_{nk+1}) < \frac{1}{2^k}$$

We obtain that

$$\sum_{k=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Therefore, $\{x_{nk}\}$ is a Cauchy sequence and so converges to some $x \in X$. By continuity of T, we have

$$\lim_{\eta\to\infty} x_{nk+1} = \lim_{\eta\to\infty} Tx_{nk} = Tx.$$

This implies that x is a fixed point of T. On the other hand, since

$\alpha(x_{nk}, x_{nk}+1) \geq \eta(x_{nk}, x_{nk}+1)$, $\forall k \in \mathbb{N}$ and $\{x_{nk}\}$ converges to x, we obtain that

$$\alpha(x_{nk}, x) \geq \eta(x_{nk}, x), \quad \forall k \in \mathbb{N}.$$

Using (ii), for each $k \in \mathbb{N}$, we have

$$d(Tx, x) \leq d(Tx, Tx_{nk}) + d(Tx_{nk}, x)$$
$$\leq \psi(d(x_{nk}, x)) + d(x_{nk}+1, x).$$

Since ψ is upper semicontinuous from the right, we obtain that

$$\limsup_{k\to\infty} \psi(d(x_{nk}, x)) \leq \psi(0) = 0.$$

By taking the limit as $k \to \infty$, this yields $d(Tx, x) = 0$ and hence $Tx = x$.

Theorem(10):- Suppose all hypotheses of Theorem (9) hold.
Assume that, for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(y, z) \geq \eta(y, z)$. Then, T has a unique fixed point.

Proof:- Assume that x and y are two fixed points of T. This implies that there exists $z \in X$ such that

$$\alpha(x, z) \geq \eta(x, z), \qquad \alpha(y, z) \geq \eta(y, z).$$

Since T is α-admissible with respect to η, for each $n \in \mathbb{N}$, we obtain that

$$\alpha(x, T^n z) \geq \eta(x, T^n z), \qquad \alpha(y, T^n z) \geq \eta(y, T^n z).$$

It follows that

$$d(x, T^{n+1}z) = d(Tx, T^{n+1}z) \leq \psi(d(x, T^n z)) < d(x, T^n z).$$

Therefore, $\{d(x, T^n z)\}$ is a nonincreasing sequence and then converges to some $c \in \mathbb{R}$. We will show that $c = 0$. Suppose that $c > 0$. Since ψ is upper semicontinuous from the right, we have

$$c = \limsup_{n \to \infty} d(x, T^{n+1}z) \leq \limsup_{n \to \infty} \psi(d(x, T^n z)) \leq \psi(c) < c,$$

which is a contradiction. It follows that $\lim_{\eta \to \infty} d(x, T^n z) = 0$,

Similarly, by the same argument, we can prove that $\lim_{\eta \to \infty} d(y, T^n z) = 0$

Since the limit of the sequence is unique, we have $x = y$.

Applying Theorems (9) and (10), we immediately obtain the following result.

Corollary(11):- Let (X, d) be a complete metric space and $\psi \in \Psi_2$. Suppose that $T : X \to X$ is an α-ψ-contractive mapping satisfying the following conditions:

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathrm{N}$ and $x_n \to x \in X$ as $n \to \infty$, and then $\alpha(x_n, x) \geq 1$ for all $n \in \mathrm{N}$;

(iv) for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Then, T has a unique fixed point.

Bhaskar and Lakshmikantham introduced the definition of coupled fixed points.

Definition (11):- Let $F : X \times X \to X$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$F(x, y) = x, \qquad F(y, x) = y.$$

Remark (12):- Let $F : X \times X \to X$ be a given mapping. Define the mapping $T : X \times X \to X \times X$ by $T(x, y) = (F(x, y), F(y, x)) \quad \forall \ (x, y) \in X \times X$.

Therefore, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T.

Theorem (13):- Let (X, d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exist $\psi \in \Psi_2$ and a function $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that

$$\alpha((x, y), (u, V))\, d(F(x, y), F(u, V)) \leq \frac{1}{2} \psi(d(x, u) + d(y, V)), \qquad \text{for all } (x, y) \in X$$

Suppose that,

(i) for all $(x, y), (u, V) \in X \times X$, one has $\alpha((x, y), (u, V)) \geq 1$ *implies* $\alpha((F(x, y), F(y, x)), (F(u, V), F(V, u))) \geq 1$;

(ii) there exists $(x_0, y_0) \in X \times X$ such that

$$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1,$$
$$\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1;$$

(iii) F is continuous.

Then, F has a coupled fixed point.

Theorem(14):- Let (X, d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exist $\psi \in \Psi_2$ and a function $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that

$$\alpha((x, y), (u, v))\, d(F(x, y), F(u, v)) \leq \frac{1}{2} \psi(d(x, u) + d(y, v)),$$

for all $(x, y), (u, v) \in X \times X$. Suppose that,

(i) for all (x, y) , (u, v) $\in X \times X$, we have

$\alpha((x, y), (u, v)) \geq 1$
$\Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$

(ii) there exists $(x_0, y_0) \in X \times X$ such that $\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$,
$\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1$;

(iii) if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$\alpha((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1$,
$\alpha((y_{n+1}, x_{n+1}), (y_n, x_n)) \geq 1$,

$x_n \to x \in X, \quad y_n \to y \in X$ as $n \to \infty$,

then
$\alpha((x_n, y_n), (x, y)) \geq 1, \alpha((y, x), (y_n, x_n)) \geq 1$

Then, F has a coupled fixed point.

Theorem(14):- Suppose that all hypotheses of Theorem 17 (resp., Theorem 18) hold. Assume that, for all (x, y), (u, v) $\in X \times X$, there exists $(z_1, z_2) \in X \times X$ such that

$\alpha((x, y), (z_1, z_2)) \geq 1$, $\alpha((z_2, z_1), (y, x)) \geq 1$, $\alpha((u, v), (z_1, z_2)) \geq 1$, $\alpha((z_2, z_1), (v, u)) \geq 1$.

Then, F has a unique coupled fixed point.

Consequences:-

We now prove the fixed point theorems in complete metric spaces and partially ordered complete metric spaces using our obtained results.

Theorem(15):- Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying

$d(Tx, Ty) \leq k\, d(x, y)$, for all $x, y \in X$, where $k \in [0, 1)$.

Then, T has a unique fixed point.

Proof. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be mappings defined by

$\alpha(x, y) = 1, \qquad \eta(x, y) = 1 \qquad \forall\, x, y \in X.$

It follows that T is α-admissible with respect to η. Suppose that ψ : $[0, +\infty) \to [0, +\infty)$ defined by $\psi(t) = kt$ for all $t \in [0, +\infty)$. This implies that ψ is upper semicontinuous from the right, $\psi(t) < t$ for all $t \in (0, +\infty)$ and $\psi(0) = 0$. Further, we can see that all assumptions in Theorem (10) are now satisfied. This completes the proof.
Theorem(16):- Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric $d \to X$ be a continuous and non-decreasing mapping with respect to $\preccurlyeq$. Assume that the following conditions hold:

(i) there exists $k \in [0, 1)$ $d(Tx, Ty) \leq k(d(x, y))$ for all x, $y \in X$. With x $\preccurlyeq$ y ;

(ii) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;

(iii) T is continuous. in X such that the metric space (X, d) is complete.

Let $T : X$ Then, T has a fixed point.

Proof:- Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ are mappings defined by

$$\alpha(x, y) = \{ 1, \text{ if } x \preccurlyeq y \text{ and } 0, \text{otherwise } \}$$

$$\eta(x, y) = \{ \tfrac{1}{2}, \text{ , if } x \preccurlyeq y \text{ and } 2, \text{otherwise } \}$$

Let $x, y \in X$ such that $\alpha(x, y) \geq \eta(x, y)$. This implies that $x \preccurlyeq y$. Since T is non-decreasing with respect to $\preccurlyeq$, we obtain that $Tx \preccurlyeq Ty$. Therefore, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. It follows that T is α-admissible with respect to η. Define a mapping $\psi : [0, +\infty) \to [0, +\infty)$ defined by $\psi(t) = kt$ for all $t \in [0, +\infty)$. We can see that $\psi \in \Psi_2$. For each $x, y \in X$ with
$\alpha(x, y) \geq \eta(x, y)$, we obtain that $x \preccurlyeq y$ and this yields

$$d(Tx, Ty) \leq k(d(x, y)) = \psi(d(x, y)) .$$

By using (ii), we have $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Hence, all assumptions in Theorem(9) are now satisfied. Thus, we obtain the desired result.

Theorem(17):- Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preccurlyeq$. Assume that the following conditions hold:

(i) there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq k(d(x, y))$ for all $x, y \in X$ with $x \preccurlyeq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$;

(iii) if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ as $n \to \infty$,

then $x_n \preccurlyeq x$ for all $n \in \mathrm{N}$.

Then, T has a fixed point.

Proof:- Suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ and $\psi : [0, +\infty) \to [0, +\infty)$ are mappings defined as in the proof of Theorem(16). Assume that $\{x_n\}$ is a sequence in X such that
$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathrm{N}$ and $x_n \to x \in X$ as $n \to \infty$.This implies that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathrm{N}$. Using (iii), this yield $x_n \preccurlyeq x$ for all $n \in \mathrm{N}$. Therefore, $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathrm{N}$. Hence, all assumptions in Theorem(9) are now satisfied.
Thus, we obtain the desired result.

Applications to Ordinary Differential Equations:-

The following ordinary differential equation is taken from Samet et al. Denote by $C([0, 1])$ the set of all continuous functions defined on $[0, 1]$ and let $d : C([0, 1]) \times C([0, 1]) \to \mathrm{R}$ be $d(x, y) = \|x - y\|_\infty = \max | x(t) - y(t) |$, for every $t \in [0, 1]$

It is well known that $(C([0, 1]), d)$ is a complete metric space. Let us consider the two-point boundary value problem of the second-order differential equation:

$$-\frac{d^2x}{dt^2} = f(t, x(t)), \quad t \in [0, 1], \quad x(0) = x(1) = 0 ,$$

where $f: [0, 1] \times \mathrm{R} \to \mathrm{R}$ is continuous. The Green function is defined by

$$G(t, s) = \{ t(1-s), \ 0 \leq t \leq s \leq 1; \text{ or } s(1-t), \ 0 \leq s \leq t \leq 1 . \}$$

Assume that the following conditions hold:

(i) there exists a function $\phi: \mathrm{R}^2 \to \mathrm{R}$ such that, for all $t \in [0, 1]$, for all $t \in [0, 1]$, for all $a, b \in \mathrm{R}$ with $\phi(a, b) \geq 0$, we have

$$| f(t, a) - f(t, b) | \leq 8 \psi (\max | a - b | , \text{ for all } a, b \in \mathrm{R}, \ \& \ \phi(a,b) \geq 0), \text{ where } \psi \in \Psi_2;$$

(ii) there exists $x_0 \in C([0,1])$ such that, for all $t \in [0, 1]$, we have

$\phi(x_0(t), \int_0^1 G(t,s) f(s, x_0(s))\, ds) \geq 0$,

(iii) for all $t \in [0, 1]$, for all $x, y \in C([0, 1])$, $\phi(x(t), y(t)) \geq 0$ implies

$\Phi(\int_0^1 G(t,s) f(s, x(s))\, ds, \int_0^1 G(t,s) f(s, y(s))\, ds) \geq 0$;

(iv) if $\{x_n\}$ is a sequence in $C([0, 1])$ such that $x_n \to x \in C([0, 1])$ and $\phi(x_n, x_{n+1}) \geq 0$, for all $n \in \mathbb{N}$, then $\phi(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.

We now prove that existence of a solution of the mentioned second-order differential equation. The idea of proving the following theorem is taken from theorem(9) but is slightly different.

Theorem(18):- Under assumptions (i) – (iv), we have a solution in $C^2([0, 1])$.

Proof:- It is well known that $x \in C^2([0, 1])$ is a solution which is equivalent to $x \in C([0, 1])$ is a solution of the integral equation.

$X(t) = \int_0^1 G(t,s) f(s, x(s))\, ds$, for every $t \in [0, 1]$.

Let $T: C([0, 1]) \to C([0, 1])$ be a mapping defined by

$Tx(t) = \int_0^1 G(t,s) f(s, x(s))\, ds$, $\forall\, t \in [0,1]$

Suppose that $x, y \in C([0, 1])$ such that $\phi(x(t), y(t)) \geq 0$ for all $t \in [0, 1]$. By applying (i), we obtain that

$| Tx(t) - Ty(t) | = | \int_0^1 G(t,s) [f(s, x(s)) - f(s, y(s))] ds |$

$\leq \int_0^1 G(t,s) | f(s, x(s)) - f(s, y(s)) | ds$

$\leq 8 (\int_0^1 G(t,s) ds (\psi (\| x - y \|_\infty))$

$\leq 8 (\sup. \int_0^1 G(t,s) ds, \forall \in [0,1]) (\psi (\| x - y \|_\infty))$

Since $\int_0^1 G(t,s) ds = -\frac{t^2}{2} + \frac{t}{2}$, $\forall \in [0,1])$, we have

Sup $\int_0^1 G(t,s) ds = \frac{1}{8}$, $\forall \in [0,1])$, If follow that

$\| Tx - Ty \|_\infty \leq \psi (\| x - y \|_\infty)$,

For each $x, y \in C([0,1])$, such that $\Phi(x(t), y(t)) \geq 0$ for all $t \in [0,1]$.

Let $\alpha, \eta : C([0,1] \times C([0,1]) \to [0, \infty]$ be mappings defined by

$\alpha(x, y) = \{ 1, \Phi(x(t), y(t) \geq 0, t \in [0,1]; \text{ or } 0, \text{ otherwise}, \}$

$\eta(x, y) = \{ ½, \Phi(x(t), y(t)) \geq 0, t \in [0,1]; \text{ or } 2, \text{ otherwise}. \}$

Let $x, y \in C([0, 1])$ s.t. $\alpha(x, y) \geq \eta(x, y)$.This implies that $\phi(x(t), y(t)) \geq 0$ for all $t \in [0, 1]$. Therefore,

Further, if $x, y \in C([0, 1])$ such that $\alpha(x, y) \geq \eta(x, y)$, then by using (iii), we have

$\phi(Tx(t), Ty(t)) \geq 0$

and this yields $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

It follows that T is α-admissible with respect to η. By (ii), there exists $x_0 \in C([0, 1])$ such that

$\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$.

Applying Theorem (9), we obtain that T has a fixed point in $C([0, 1])$.

Conclusion:- In this paper, we prove some applications of fixed point theorem's with respect to a non-linear differential equations.

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